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Computation of Detour D-Index and Average Detour D-Distance of Specific Graphs

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ABSTRACT

The length of the longest $u-v$ path in a connected graph Λ is the detour distance $D(u,v)$ for any two different vertices u and v. The detour D-distance denoted by $D^D(u,v)$ and defined by $D^D(u,v)=max\,\{l^D(P)\}$, where the maximum is taken over all u–v paths P in Λ. The detour $D-$index of Λ is defined by $W_D^D(\Lambda)=\frac{1}{2}\sum_{u,v\in V(\Lambda)}D^D(u,v)=\sum_{\{u,v\}\subseteq V(\Lambda)}D^D(u,v)$. The detour D-index of several graphs, including the French windmill, Kulli-wheel windmill, lollipop, general barbell, and general modified barbell graphs, is studied and obtained in this paper. In addition, these graphs' average detour D-distance will be determined.

Keywords: Detour distance, Detour D-index, Average detour D-distance.

1. Introduction

By a graph Λ, we mean a nontrivial, finite, undirected, simple, and connected graph. We refer to[1, 2] for any unexplained notations and terminology. A graph $\Lambda=(V,\ E)$, where $V(\Lambda)$ or V is the vertex set of Λ and $E(\Lambda)$ or E is the edge set of Λ. A (p,q) graph Λ has order p and size q where $p=|V(\Lambda)|$ is the order of Λ and $q=|E(\Lambda)|$ its size. Degree of a vertex v, denoted by $deg_\Lambda(v)$ or more simply deg (v) is, refers to the number of edges incident to the vertex v.The concept of distance is one of the essential concepts in the study of graphs. The standard or usual distance $d(u,v)$ between any two arbitrary vertices u and v of a graph Λ is the length of the minimum path connecting u and v. Many researchers have determined different concepts of distances as well as ordinary distance, such as Steiner distance, width distance, signed distance and so forth. In addition to the usual distance $d(u,v)$ we have detour distance $D(u,v)$ introduced by Chartrand et al.,[3], which is the length of the longest path between distinct vertices u and v. They discussed the detour distance and its related properties. The restricted detour distance between vertices u and v in a graph Λ, denoted by $D^*(u,v)$ or $D^*_\Lambda(u,v)$ is the length of a longest u–v path P such that $<V(P)>=P$ which introduced in[4, 5]. Kathiresan and Marimuthu[6] have introduced the concepts of superior distance and signal distance. In some of the earlier distances, only the lengths of various paths were considered. However, Babu and Varma[7, 8] have introduced the concept of D–distance, $d^D(u,v)$ between the vertices considering not only the path length but alsothe degree of all the vertices present in the path between u and v, the D-length of a $u-v$ path P is defined as the sum of the length of the path P together with the degrees of u and v and the degrees of all intermediate vertices of the path P. The idea of detour D-distance (DDD) and some work-related was first introduced in[9]. For u and v, the detour D-distance is depending on the detour distance between u and v and the degree of vertices that lie on $u-v$ path. For any connected graph the D-distance and detour D-distance, are metric on the set of vertices of Λ which is proved in[7, 10].

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In graph theory, the average distance is a crucial parameter, serving as a key parameter in analytic networks. This metric's significance lies in the fact that the time required for performance is directly related to the distance between two points within the network [11].

In this article, we obtain detour D–index (DDI) for various specific graphs, such as lollipop, general barbell, general modified barbell, French windmill, and Kulli-wheel windmill graphs. Also, we compute the average DDD of these graphs.

Definition 1.1:[9] For any two vertices u and v in Λ, the $D-$length of a $u-v$ path P is defined as $l^D(P)=l(P)+deg(u)+deg(v)+\sum deg\ (\omega)$, where the sum runs over all intermediate vertices ω of P and $l(P)$ is the length of the path.

Definition 1.2:[9] The DDD between two different vertices u and v in Λ is defined as $D^D(u,v)=max\,\{l^D(P)\}$, where the maximum is taken over all $u-v$ paths P in Λ. In other words,

$$D^D(u,v)=\begin{cases}max\{l^D(P)\} & if\ u\neq v\\ 0 & if\ u=v.\end{cases}$$

The DDI of Λ denoted by $W_D^D(\Lambda)$, is the sum of all DDDs between any two distinct vertices of Λ, that is $W_D^D(\Lambda) = \sum_{\{u,v\}\subseteq V(\Lambda)} D^D(u,v)$.

Definition 1.3:[11] Let Λ be a connected graph, of order ϱ, then the average DDD of Λ indicated by $\mu_D^D(\Lambda)$ and defined as $\mu_D^D(\Lambda) = \frac{1}{\binom{\varrho}{2}}\sum_{\{u,v\}\subseteq V(\Lambda)} D^D(u,v)$.

2. Detour D −index of some graphs

This section provides the DDD and DDI for different graph families. Also, their average DDDs are computed. We start with lollipop graph $L_{\varrho,\xi}$.

Definition 2.1:[12] The **lollipop graph $L_{\varrho,\xi}$** is a graph generated by joining a complete graph $K_\varrho, \varrho \geq 3$, and a path graph $P_\xi, \xi \geq 2$ with a bridge. The graph $L_{\varrho,\xi}$ is depicted in Figure 1.

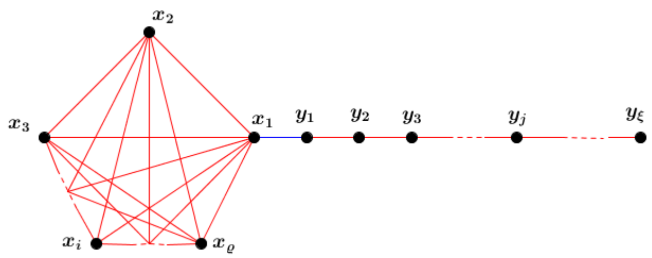

Figure 1: Lollipop graph $\boldsymbol{L_{\varrho,\xi}}$

Theorem 2.1. The DDI of the lollipop graph $L_{\varrho,\xi}$, for all $\varrho \geq 3$, $\xi \geq 2$, is

$$W_D^D(L_{\varrho,\xi}) = \binom{\varrho}{2}\varrho^2 + 3\binom{\xi}{2} + \frac{(\xi-1)^2(\xi+4)}{2} + (\xi-1)(\varrho+1) + 3\xi + \varrho - 1 + \frac{(\varrho-1)}{2}[(2\varrho-2)(\varrho-2) + 6(\varrho+\xi-1) + (\xi-1)(2\varrho^2+3\xi)].$$

Proof: For any two distinct vertices of lollipop graph(see Figure 1), the following four main cases are considered.

Case 1. Any two distinct vertices in a complete graph $K_\varrho, \varrho \geq 3$.

Due to the fact that the DDI for the complete graph which is equal to $\binom{\varrho}{2}(\varrho^2-1)$. Thus, the DDI for the complete graph K_ϱ^l in the lollipop graph is equal to $\binom{\varrho}{2}(\varrho^2-1) + \binom{\varrho}{2} = \binom{\varrho}{2}\varrho^2$.

Case 2. Any two distinct vertices in a path graph $P_\xi, \xi \geq 2$.

Due to the fact that the DDI for the path graph which is equal to $\frac{(\xi-1)^2(\xi+4)}{2}$. Thus, the DDI for the path graph P_ξ^l in the lollipop graph is equal to $\frac{(\xi-1)^2(\xi+4)}{2} + (\xi-1)$.

Case 3. The DDD between the vertex x_1 and all the other vertices in the path graph P_ξ is given by the following two subcases:

i. $D^D(x_1, y_j) = 3j + \varrho$, for $j = 1, 2, \ldots, \xi-1$.

Thus,

$$\sum_{j=1}^{\xi-1}(3j+\varrho) = 3\left(\frac{\xi(\xi-1)}{2}\right) + \varrho(\xi-1) = (\xi-1)\left(\frac{3\xi+2\varrho}{2}\right).$$

ii. $D^D(x_1, y_\xi) = 3\xi + \varrho - 1$.

Case 4. The DDD between vertices x_i and y_j, for $i = 2, 3, \ldots, \varrho$ and $j = 1, 2, \ldots, \xi$ is given by the following two subcases:

i. $D^D(x_i, y_\xi) = (\varrho-1)(\varrho-2) + 3(\varrho+\xi-1)$ for $i = 2, 3, \ldots, \varrho$.

Thus,

$$\sum_{i=2}^{\varrho}\big((\varrho-1)(\varrho-2) + 3(\varrho+\xi-1)\big) = (\varrho-1)\big((\varrho-1)(\varrho-2) + 3(\varrho+\xi-1)\big).$$

ii. $D^D(x_i, y_j) = (\varrho-1)(\varrho-2) + 3(\varrho+j) - 2$.

Thus,

$$\begin{aligned}\sum_{j=1}^{\xi-1}\sum_{i=2}^{\varrho}\big((\varrho-1)(\varrho-2) + 3(\varrho+j) - 2\big) &= (\varrho-1)\sum_{j=1}^{\xi-1}\big((\varrho-1)(\varrho-2) + 3(\varrho+j) - 2\big)\\ &= (\varrho-1)\left[(\xi-1)\big((\varrho-1)(\varrho-2) + 3\varrho - 2\big) + \frac{3\xi(\xi-1)}{2}\right] = \frac{(\varrho-1)(\xi-1)}{2}(2\varrho^2+3\xi).\end{aligned}$$

Hence,

$$\begin{aligned}W_D^D(L_{\varrho,\xi}) = \sum_{\{u,v\}\subseteq V(L_{\varrho,\xi})} D^D(u,v) &= W_D^D(K_\varrho^l) + W_D^D(P_\xi^l) + \sum_{j=1}^{\xi-1} D^D(x_1, y_j) + D^D(x_1, y_\xi)\\ &+ \sum_{i=2}^{\varrho} D^D(x_i, y_\xi) + \sum_{j=1}^{\xi-1}\sum_{i=2}^{\varrho} D^D(x_i, y_j)\\ &= \binom{\varrho}{2}\varrho^2 + \frac{(\xi-1)^2(\xi+4)}{2} + (\xi-1)\\ &+ (\xi-1)\left(\frac{3\xi+2\varrho}{2}\right) + 3\xi + \varrho - 1\\ &+ (\varrho-1)\big((\varrho-1)(\varrho-2) + 3(\varrho+\xi-1)\big)\\ &+ \frac{(\varrho-1)(\xi-1)}{2}(2\varrho^2+3\xi)\\ &= \binom{\varrho}{2}\varrho^2 + 3\binom{\xi}{2} + \frac{(\xi-1)^2(\xi+4)}{2}\\ &+ (\xi-1)(\varrho+1) + 3\xi + \varrho - 1\\ &+ \frac{(\varrho-1)}{2}[(2\varrho-2)(\varrho-2) + 6(\varrho+\xi-1)\\ &+ (\xi-1)(2\varrho^2+3\xi)]. \blacksquare\end{aligned}$$

Corollary 2.2. For the lollipop graph $L_{\varrho,\xi}$ for all $\varrho \geq 3,\ \xi \geq 2,$ we have the average DDD

$$\mu_D^D(L_{\varrho,\xi}) = \frac{\xi^3+(3\varrho+2)\xi^2+(2\varrho^3-2\varrho^2+5\varrho-5)\xi+(\varrho-1)(\varrho^3-2)}{(\varrho+\xi)(\varrho+\xi-1)}.$$

Proof:

$$\mu_D^D(L_{\varrho,\xi}) = \frac{W_D^D(L_{\varrho,\xi})}{\binom{\varrho+\xi}{2}} = \frac{2W_D^D(L_{\varrho,\xi})}{(\varrho+\xi)(\varrho+\xi-1)},$$

where,

$$\begin{aligned} W_D^D(L_{\varrho,\xi}) &= \left(\binom{\varrho}{2}\varrho^2+3\binom{\xi}{2}+\frac{(\xi-1)^2(\xi+4)}{2}+(\xi-1)(\varrho+1)+3\xi \right.\\ &\quad \left. +\varrho-1\right) \\ &\quad +\left(\frac{(\varrho-1)}{2}[(2\varrho-2)(\varrho-2)+6(\varrho+\xi-1)+(\xi-1)(2\varrho^2+3\xi)]\right) \\ &= \left(\frac{2\varrho^2\binom{\varrho}{2}+6\binom{\xi}{2}+(\xi-1)^2(\xi+4)+2\big((\xi-1)(\varrho+1)+3\xi+\varrho-1\big)}{2}\right) \\ &\quad +\left(\frac{(\varrho-1)(2\xi\varrho^2+3\xi^2+3\xi-2)}{2}\right) \\ &= \left(\frac{\xi^3+5\xi^2+2\xi(\varrho-1)+\varrho^3(\varrho-1)}{2}\right) \\ &\quad +\left(\frac{(\varrho-1)(2\xi\varrho^2+3\xi^2+3\xi-2)}{2}\right) \\ &= \frac{\xi^3+(3\varrho+2)\xi^2+(2\varrho^3-2\varrho^2+5\varrho-5)\xi+(\varrho-1)(\varrho^3-2)}{2}. \end{aligned}$$

Thus,

$$\mu_D^D(L_{\varrho,\xi}) = \frac{\xi^3+(3\varrho+2)\xi^2+(2\varrho^3-2\varrho^2+5\varrho-5)\xi+(\varrho-1)(\varrho^3-2)}{(\varrho+\xi)(\varrho+\xi-1)}. \blacksquare$$

Remark 2.1: For $\varrho = \xi$,

$$\mu_D^D(L_{\varrho,\varrho}) = \frac{3\varrho^4+\varrho^3+7\varrho^2-7\varrho+2}{4\varrho^2-2\varrho}.$$

Definition 2.2:[13] A **general barbell graph** $\boldsymbol{B_{\varrho,\xi}}$ is a graph generated by joining two complete graphs K_ϱ, and K_ξ, $\varrho,\xi \geq 3$, by a bridge, as a particular case, if $\varrho = \xi$, then the resulting graph is called barbell graph, denoted by B_ϱ .

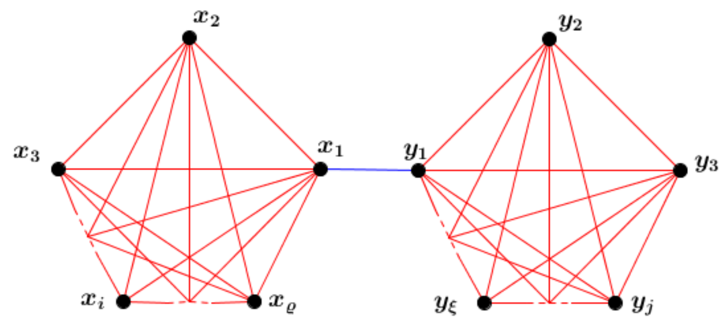

Figure 2: General barbell graph $\boldsymbol{B_{\varrho,\xi}}$

Theorem 2.3. The DDI of general barbell graph $B_{\varrho,\xi}$, for all $\varrho,\xi \geq 3$, is

$$W_D^D(B_{\varrho,\xi}) = \binom{\varrho}{2}\varrho^2+\binom{\xi}{2}\xi^2+(\varrho+\xi+1)+\wp_{\varrho,\xi}+\mathfrak{H}_{\varrho,\xi},$$

where

$$\begin{aligned}\wp_{\varrho,\xi} &= (\xi-1)[3\xi+\varrho-1+(\xi-1)(\xi-2)] \\ &\quad +(\varrho-1)[3\varrho+\xi-1+(\varrho-1)(\varrho-2)],\end{aligned}$$

and

$$\begin{aligned}\mathfrak{H}_{\varrho,\xi} &= (\varrho-1)(\xi-1)[3(\varrho+\xi-1)+(\varrho-1)(\varrho-2) \\ &\quad +(\xi-1)(\xi v2)].\end{aligned}$$

Proof: For any two distinct vertices of general barbell graph $B_{\varrho,\xi}$, $\varrho,\xi \geq 3$, see Figure 2, we have the following main cases.

Case 1. Any two distinct vertices in a complete graph $K_\varrho, \varrho \geq 3$.

Due to the fact that the DDI for the complete graph $K_\varrho, \varrho \geq 3$ is equal to $\binom{\varrho}{2}(\varrho^2-1)$. Thus, DDI for the complete graph K_ϱ^{gb} in the general barbell graph is equal to $\binom{\varrho}{2}(\varrho^2-1)+\binom{\varrho}{2}=\binom{\varrho}{2}\varrho^2$. And, the DDI for the complete graph $K_\xi^{gb}, \xi \geq 3$ in the general barbell graph is equal to $\binom{\xi}{2}(\xi^2-1)+\binom{\xi}{2} = \binom{\xi}{2}\xi^2$.

Case 2. The DDD of the vertices in different parts is given by the following subcases:

i. $D^D(x_1,y_1) = \varrho+\xi+1.$

ii. $D^D(x_1,y_j) = 3\xi+\varrho-1+(\xi-1)(\xi-2),$ for $j = 2, 3, \ldots, \xi.$

Thus,

$$\sum_{j=2}^{\xi} D^D(x_1,y_j) = (\xi-1)(3\xi+\varrho-1+(\xi-1)(\xi-2)).$$

iii. $D^D(x_i,y_1) = 3\varrho+\xi-1+(\varrho-1)(\varrho-2),$ for $i = 2, 3, \ldots, \varrho.$

Thus,

$$\sum_{i=2}^{\varrho} D^D(x_i, y_1) = (\varrho-1)(3\varrho + \xi - 1 + (\varrho-1)(\varrho-2)).$$

iv. $D^D(x_i, y_j) = 3(\varrho + \xi - 1) + (\varrho-1)(\varrho-2) + (\xi-1)(\xi-2)$, for $i = 2, 3, \ldots, \varrho$ and $j = 2, 3, \ldots, \xi$.

Thus,

$$\sum_{j=2}^{\xi}\sum_{i=2}^{\varrho} D^D(x_i, y_j) = (\varrho-1)(\xi-1)[3(\varrho + \xi - 1) + (\varrho-1)(\varrho-2) + (\xi-1)(\xi-2)].$$

Hence,

$$\begin{aligned}
W_D^D(B_{\varrho,\xi}) &= \sum_{\{u,v\}\subseteq V(B_{\varrho,\xi})} D^D(u,v) \\
&= W_D^D(K_\varrho^{gb}) + W_D^D(K_\xi^{gb}) + D^D(x_1, y_1) \\
&+ \sum_{j=2}^{\xi} D^D(x_1, y_j) + \sum_{i=2}^{\varrho} D^D(x_i, y_1) \\
&+ \sum_{j=2}^{\xi}\sum_{i=2}^{\varrho} D^D(x_i, y_j) \\
&= \binom{\varrho}{2}\varrho^2 + \binom{\xi}{2}\xi^2 + (\varrho + \xi + 1) \\
&+ (\xi-1)(3\xi + \varrho - 1 + (\xi-1)(\xi-2)) \\
&+ (\varrho-1)(3\varrho + \xi - 1 + (\varrho-1)(\varrho-2)) \\
&+ (\varrho-1)(\xi-1)[3(\varrho + \xi - 1) + (\varrho-1)(\varrho-2) \\
&+ (\xi-1)(\xi-2)] \\
&= \binom{\varrho}{2}\varrho^2 + \binom{\xi}{2}\xi^2 + (\varrho + \xi + 1) + \wp_{\varrho,\xi} \\
&+ \mathfrak{H}_{\varrho,\xi},
\end{aligned}$$

where

$$\wp_{\varrho,\xi} = (\xi-1)[3\xi + \varrho - 1 + (\xi-1)(\xi-2)] + (\varrho-1)[3\varrho + \xi - 1 + (\varrho-1)(\varrho-2)],$$

and

$$\mathfrak{H}_{\varrho,\xi} = (\varrho-1)(\xi-1)[3(\varrho + \xi - 1) + (\varrho-1)(\varrho-2) + (\xi-1)(\xi-2)]. \blacksquare$$

Corollary 2.4. For the general barbell graph $B_{\varrho,\xi}$ for all $\varrho, \xi \geq 3$, we have the average DDD

$$\mu_D^D(B_{\varrho,\xi}) = \frac{\xi^4 + (2\varrho-1)\xi^3 - 2\varrho\xi^2 + 2(\varrho^3 - \varrho^2 + 3\varrho)\xi + \varrho^3(\varrho-1)}{(\varrho + \xi)(\varrho + \xi - 1)}.$$

Proof:

$$\mu_D^D(B_{\varrho,\xi}) = \frac{W_D^D(B_{\varrho,\xi})}{\binom{\varrho+\xi}{2}} = \frac{2W_D^D(B_{\varrho,\xi})}{(\varrho + \xi)(\varrho + \xi - 1)},$$

where,

$$\begin{aligned}
W_D^D(B_{\varrho,\xi}) &= \binom{\varrho}{2}\varrho^2 + \binom{\xi}{2}\xi^2 + (\varrho + \xi + 1) + \wp_{\varrho,\xi} + \mathfrak{H}_{\varrho,\xi} \\
&= \frac{\varrho^3(\varrho-1) + \xi^3(\xi-1) + 2(\varrho + \xi + 1) + (2\xi - 2)(3\xi + \varrho - 1 + (\xi-1)(\xi}{2} \\
&+ \frac{(2\varrho-2)(3\varrho + \xi - 1 + (\varrho-1)(\varrho-2)) + (2\varrho-2)(\xi-1)[3(\varrho + \xi - 1) + (\varrho-}{2} \\
&+ \frac{(\xi-1)(\xi-2)}{2} \\
&= \frac{\xi^4 + (2\varrho-1)\xi^3 - 2\varrho\xi^2 + 2(\varrho^3 - \varrho^2 + 3\varrho)\xi + \varrho^3(\varrho-1)}{2}.
\end{aligned}$$

Thus,

$$\mu_D^D(B_{\varrho,\xi}) = \frac{\xi^4 + (2\varrho-1)\xi^3 - 2\varrho\xi^2 + 2(\varrho^3 - \varrho^2 + 3\varrho)\xi + \varrho^3(\varrho-1)}{(\varrho + \xi)(\varrho + \xi - 1)}. \blacksquare$$

Remark 2.2: For $\varrho = \xi$,

$$\mu_D^D(B_\varrho) = \frac{3(\varrho^3 - \varrho^2 + \varrho)}{2\varrho - 1}.$$

Definition 2.3:[14] A **French windmill graph F_ς^ξ** is the graph create by taking $\xi \geq 2$ copies of the complete graph $k_\varsigma, \varsigma \geq 2$, with a vertex say v_0 in common, which has $\xi(\varsigma-1) + 1$ vertices and $\xi\binom{\varsigma}{2}$ edges. The graph of F_ς^ξ is depicted in Figure 3.

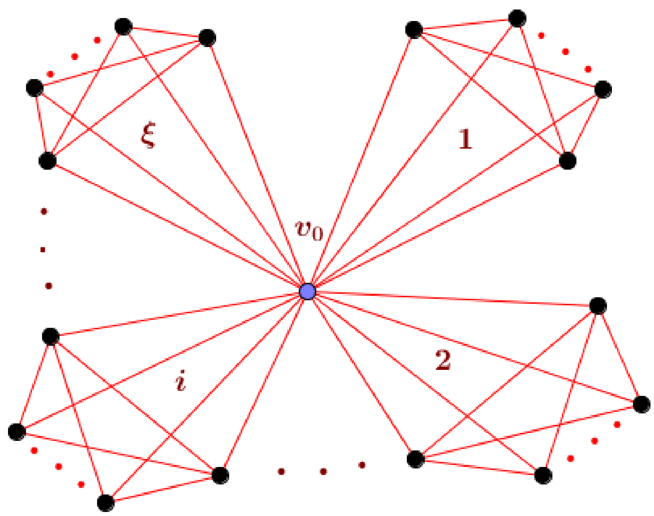

Figure 3. French windmill graph F_ς^ξ

Theorem 2.5. For a French windmill graph F_ς^ξ, for all $\xi, \varsigma \geq 2$, we have

$$W_D^D(F_\varsigma^\xi) = \xi\binom{\varsigma}{2}(\varsigma^2 - 1 + (\varsigma-1)(\xi-1)) + \binom{\xi}{2}(\varsigma-1)^3(2\varsigma + \xi).$$

Proof: For the prove, we consider two parts. For any two distinct vertices in F_ς^ξ, we have

Case 1. The DDI for all copies of the complete graph $K_\varsigma, \varsigma \geq 2$.

From the reason that the DDI of the complete graph K_ς is equal to $\binom{\varsigma}{2}(\varsigma^2-1)$, the DDI for a complete graph K_ς has the the same value $\binom{\varsigma}{2}(\varsigma^2-1)$, with the including degree of v_0 for each pair $\binom{\varsigma}{2}$ in each copies K_ς which is equal to $(\varsigma-1)(\xi-1)$. Thus, the DDI for each copy of the complete graph K_ς^F in French windmill graph is equal to $\binom{\varsigma}{2}(\varsigma^2-1)+\binom{\varsigma}{2}(\varsigma-1)(\xi-1))=\binom{\varsigma}{2}(\varsigma^2-1+(\varsigma-1)(\xi-1))$. That is $W_D^D(K_\varsigma^F)=\binom{\varsigma}{2}\left(\varsigma^2-1+(\varsigma-1)(\xi-1)\right)$, for each copy of the graph K_ς^F, $\varsigma \geq 2$. As we know, there are ξ -copies of the complete graph K_ς^F, and the DDI for the entire copies is equal to $\xi\binom{\varsigma}{2}\left(\varsigma^2-1+(\varsigma-1)(\xi-1)\right)$.

Case 2. The DDD for the remaining vertices.

Since we know that all the remaining vertices are non-adjacent, also from the fact that the symmetry between copies will give the same value for each remaining non-adjacent vertices which is enough to find the DDD of only one order pair. So, the procedure will be completed after finding DDD of only one order pair. Therefore, for any non-adjacent vertices u and v, we have

$$\begin{aligned} D^D(u,v) &= (\varsigma-1+\varsigma-1)+(\varsigma-1)+(\varsigma-1) \\ &\quad+\left(\xi(\varsigma-1)+2(\varsigma-1)(\varsigma-2)\right) \\ &= (\varsigma-1)(2\varsigma+\xi). \end{aligned}$$

Also, there are $(\varsigma-1)^2$ non-adjacent remaining vertices between any two copies of the complete graph K_ς. Thus, due to the symmetric property, we obtain that the total DDD of the remaining non-adjacent vertices is $(\varsigma-1)^2(\varsigma-1)(2\varsigma+\xi)=(\varsigma-1)^3(2\varsigma+\xi)$. For that reason, we have a total DDD equal to $(\varsigma-1)^3(2\varsigma+\xi)$ between any two copies of K_ς. Thus, the total DDD for the remaining non-adjacent vertices can be expressed as their combination of copies which is given below. Therefore, for all the remaining non-adjacent vertices u and v, we have,

$$D^D(u,v)=(\xi-i)(\varsigma-1)^3(2\varsigma+\xi), \qquad \text{for } i=1,2,\dots,\xi-1.$$

Thus,

$$\begin{aligned} \sum_{i=1}^{\xi-1} D^D(u,v) &= \sum_{i=1}^{\xi-1}(\xi-i)(\varsigma-1)^3(2\varsigma+\xi) \\ &= (\varsigma-1)^3(2\varsigma+\xi)\sum_{i=1}^{\xi-1}(\xi-i) \\ &= (\varsigma-1)^3(2\varsigma+\xi)\left[\xi(\xi-1)-\frac{\xi(\xi-1)}{2}\right] \\ &= \binom{\xi}{2}(\varsigma-1)^3(2\varsigma+\xi). \end{aligned}$$

Hence,

$$\begin{aligned} W_D^D\left(F_\varsigma^\xi\right) &= \xi\binom{\varsigma}{2}\left(\varsigma^2-1+(\varsigma-1)(\xi-1)\right) \\ &\quad+\binom{\xi}{2}(\varsigma-1)^3(2\varsigma+\xi). \blacksquare \end{aligned}$$

Corollary 2.6. For the French windmill graph F_ς^ξ for all $\varsigma,\xi \geq 2$, we have the average DDD

$$\mu_D^D\left(F_\varsigma^\xi\right) = \frac{\left(\sqrt{\xi}\,\varsigma-\sqrt{\xi}\right)^2\left((\varsigma-1)\xi^2+(2\varsigma^2-2\varsigma+1)\xi+\varsigma(2-\varsigma)\right)}{(\xi(\varsigma-1))^2+\xi(\varsigma-1)}.$$

Proof:

$$\mu_D^D\left(F_\varsigma^\xi\right)=\frac{W_D^D\left(F_\varsigma^\xi\right)}{\binom{\xi(\varsigma-1)+1}{2}}=\frac{2W_D^D\left(F_\varsigma^\xi\right)}{(\xi(\varsigma-1)+1)(\xi(\varsigma-1))}$$

where,

$$\begin{aligned} W_D^D\left(F_\varsigma^\xi\right) &= \xi\binom{\varsigma}{2}\left(\varsigma^2-1+(\varsigma-1)(\xi-1)\right)+\binom{\xi}{2}(\varsigma-1)^3(2\varsigma+\xi) \\ &= \frac{\left(\sqrt{\xi}\,\varsigma-\sqrt{\xi}\right)^2\left(\varsigma(\varsigma+\xi)+(\xi-1)(\varsigma-1)(2\varsigma+\xi)\right)}{2} \\ &= \frac{\left(\sqrt{\xi}\,\varsigma-\sqrt{\xi}\right)^2\left((\varsigma-1)\xi^2+(2\varsigma^2-2\varsigma+1)\xi+\varsigma(2-\varsigma)\right)}{2}. \end{aligned}$$

Thus,

$$\mu_D^D\left(F_\varsigma^\xi\right) = \frac{\left(\sqrt{\xi}\,\varsigma-\sqrt{\xi}\right)^2\left((\varsigma-1)\xi^2+(2\varsigma^2-2\varsigma+1)\xi+\varsigma(2-\varsigma)\right)}{(\xi(\varsigma-1))^2+\xi(\varsigma-1)}. \blacksquare$$

Remark 2.3: For $\varsigma=\xi$,

$$\mu_D^D\left(F_\varsigma^\varsigma\right)=\frac{2\varsigma^2(\varsigma-1)+3\varsigma(\varsigma-1)^3}{\varsigma(\varsigma-1)+1}.$$

Definition 2.4:[12] Let $W_\varsigma, \varsigma \geq 4$ be a wheel graph, the **kulli-wheel windmill graph** $W_{\varsigma+1}^\xi$ is the graph generated by taking $\xi, \xi \geq 2$ copies $W_\varsigma+K_1$, $\varsigma \geq 4$, and K_1 is the complete graph of order 1. With a vertex v_0 of K_1 in common. The graph $W_{\varsigma+1}^\xi$ is depicted in Figure 4.

It is clear that $\left|V(W_{\varsigma+1}^\xi)\right|=\xi\varsigma+1$, $\left|E(W_{\varsigma+1}^\xi)\right|=\xi(3\varsigma-2)$.

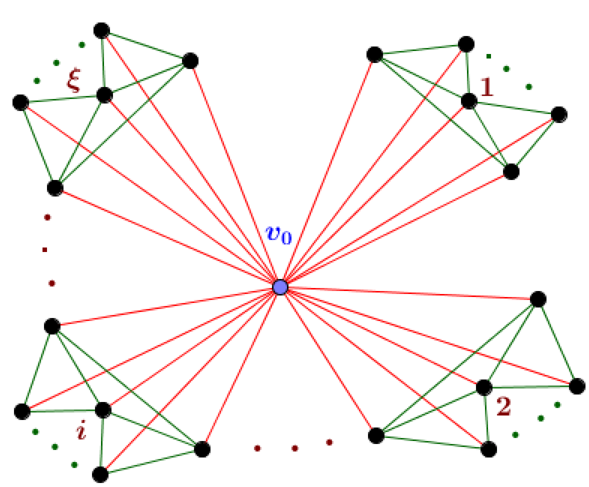

Figure 4. Kulli-Wheel Windmill Graph $W_{\varsigma+1}^\xi$

Theorem 2.7. For Kulli-wheel Windmill Graph $W^{\xi}_{\varsigma+1}$, for all $\xi \geq 2,\ \varsigma \geq 4$ we have

$$W_D^D\left(W^{\xi}_{\varsigma+1}\right) = \xi\left(\varsigma + \binom{\varsigma}{2}\right)(5(\varsigma-1) + \xi\varsigma + \varsigma + 1) + \varsigma^2\binom{\xi}{2}(\varsigma(\xi+12)-8).$$

Proof: The prove divides into two cases. For any two distinct vertices in $W^{\xi}_{\varsigma+1}$.

Case 1. The DDI for all copies of the graph $W_{\varsigma} + K_1,\ \varsigma \geq 4$.

The reason that the DDI of wheel graph W_{ς} is equal to $\binom{\varsigma}{2}(5(\varsigma-1))$, thus, DDI for wheel graph W_{ς} in $W_{\varsigma} + K_1$ has the same value $\binom{\varsigma}{2}(5(\varsigma-1))$, with the including degree of v_0 with an included edge, and degree for each pair $\binom{\varsigma}{2}$ in each copies W_{ς} which is equal to $\xi\varsigma + 1 + \varsigma$. Thus, the DDI for each copy of the wheel graph W_{ς} is equal to $\binom{\varsigma}{2}(5(\varsigma-1)) + \binom{\varsigma}{2}(\xi\varsigma + 1 + \varsigma)$, which is also equal to $\binom{\varsigma}{2}(5(\varsigma-1) + \xi\varsigma + \varsigma + 1)$. Know from $W_{\varsigma} + K_1$; we have a vertex v_o adjacent for all ς-vertices in W_{ς}, which means we have ς-new order pairs for each copy. Thus, we have ς-new DDD in $W_{\varsigma} + K_1$, so beyond doubt, all ς-new DDDs equal to $5(\varsigma-1) + \xi\varsigma + \varsigma + 1$. Thus, the DDI for each copy of the graph $W_{\varsigma} + K_1$ is equal to $(\varsigma + \binom{\varsigma}{2})(5(\varsigma-1) + \xi\varsigma + \varsigma + 1)$. In other word $W_D^D(W_{\varsigma} + K_1) = \left(\varsigma + \binom{\varsigma}{2}\right)(5(\varsigma-1) + \xi\varsigma + \varsigma + 1)$, for each copy of the graph $W_{\varsigma} + K_1,\ \varsigma \geq 4$. As we know, there are ξ-copies of $W_{\varsigma} + K_1$, and the DDI for the entire copies is equal to $\xi\left(\varsigma + \binom{\varsigma}{2}\right)(5(\varsigma-1) + \xi\varsigma + \varsigma + 1)$.

Case 2. The DDD for the remaining vertices

Since we know that all the remaining vertices are non-adjacent, also from the fact that the symmetry between copies will give the same value for each remaining non-adjacent vertices which is enough to find the DDD of only one order pair. So, the remaining procedure will be completed after finding DDD of only one order pair. Therefore, for non-adjacent vertices u and v, we have

$$D^D(u,v) = (\varsigma + \varsigma) + (\varsigma) + (\varsigma) + (\xi\varsigma + 2(\varsigma-1)(4)) = \varsigma(\xi+12) - 8.$$

Also, there are ς^2 non-adjacent remaining vertices between any two copies of $W_{\varsigma} + K_1$. Thus, due to the symmetric property, we obtain that the DDD of the remaining non-adjacent vertices are equal to $\varsigma^2(\varsigma(\xi+12)-8)$. For this reason, we have a total DDD equal to $\varsigma^2(\varsigma(\xi+12)-8)$ between any two copies of $W_{\varsigma} + K_1$. Thus, the total DDD for the remaining non-adjacent vertices can be expressed as their combination of copies which given below.

For all the remaining non-adjacent vertices u and v, we have,

$$D^D(u,v) = (\xi - i)\ \varsigma^2(\varsigma(\xi+12)-8), \qquad \text{for } i = 1,2,\dots,\xi-1.$$

Thus,

$$\sum_{i=1}^{\xi-1} D^D(u,v) = \sum_{i=1}^{\xi-1}(\xi-i)\ \varsigma^2(\varsigma(\xi+12)-8)$$
$$= \varsigma^2(\varsigma(\xi+12)-8)\sum_{i=1}^{\xi-1}(\xi-i) =$$
$$= \varsigma^2(\varsigma(\xi+12)-8)[\xi(\xi-1) - \frac{\xi(\xi-1)}{2}]$$
$$= \binom{\xi}{2}\ \varsigma^2(\varsigma(\xi+12)-8).$$

Hence,

$$W_D^D\left(F^{\xi}_{\varsigma}\right) = \xi\left(\varsigma + \binom{\varsigma}{2}\right)(5(\varsigma-1) + \xi\varsigma + \varsigma + 1) + \varsigma^2\binom{\xi}{2}(\varsigma(\xi+12)-8). \blacksquare$$

Corollary 2.8. For kulli-wheel windmill graph $W^{\xi}_{\varsigma+1}$ for all $\varsigma \geq 4,\ \xi \geq 2,$ we have the average DDD

$$\mu_D^D\left(W^{\xi}_{\varsigma+1}\right) = \frac{\xi\varsigma\left((\xi^2 + 12\xi - 6)\varsigma^2 + (-7\xi + 10)\varsigma - 4\right)}{\xi^2\varsigma^2 + \xi\varsigma}.$$

Proof:

$$\mu_D^D\left(W^{\xi}_{\varsigma+1}\right) = \frac{W_D^D\left(W^{\xi}_{\varsigma+1}\right)}{\binom{\xi\varsigma+1}{2}} = \frac{2W_D^D\left(W^{\xi}_{\varsigma+1}\right)}{(\xi\varsigma+1)(\xi\varsigma)}$$

where,

$$W_D^D\left(W^{\xi}_{\varsigma}\right) = \xi\left(\varsigma + \binom{\varsigma}{2}\right)(5(\varsigma-1) + \xi\varsigma + \varsigma + 1) + \varsigma^2\binom{\xi}{2}(\varsigma(\xi+12)-8)$$
$$= \frac{\xi\varsigma}{2}\left((\xi^2 + 12\xi - 6)\varsigma^2 + (-7\xi + 10)\varsigma - 4\right).$$

Thus,

$$\mu_D^D\left(W^{\xi}_{\varsigma+1}\right) = \frac{\xi\varsigma\left((\xi^2 + 12\xi - 6)\varsigma^2 + (-7\xi + 10)\varsigma - 4\right)}{\xi^2\varsigma^2 + \xi\varsigma}. \blacksquare$$

Remark 2.4: For $\varsigma = \xi$,

$$\mu_D^D\left(W^{\varsigma}_{\varsigma+1}\right) = \frac{\varsigma^4 + 12\varsigma^3 - 13\varsigma^2 + 10\varsigma - 4}{\varsigma^2 + 1}.$$

From Definitions 2.3 and 2.4, we note that the kulli-wheel windmill graph $W^{\xi}_{\varsigma+1}$ for all $\xi \geq 2$ and $\varsigma = 4$ is isomorphic to the French windmill graph F^{ξ}_5 for all $\xi \geq 2$, that is Theorem 2.7 is a special case of Theorem 2.5. Using this fact, we get the following result.

Corollary 2.9. The detour D-index of the kulli-wheel windmill graph W^{ξ}_{4+1} and French windmill graph F^{ξ}_5 for all $\xi \geq 2$, are equal. $\blacksquare$

Conclusions

The topological indices for the detour D-distance presented by researchers Rao and Varma[9] are more complex than the detour distance and the D-distance in finding a relationship between them and some chemical properties, such as boiling points and melting points, for graphs similar to chemical compounds, due to the effect of the degrees of the vertices on the algebraic quantities, as well as the detour distance between any two vertices.

In addition, we note that the indices topological and average of the detour distance and D-distance are less than the detour D-distance.

Conflict of interests

None

Author contribution

The authors of this work have made equal contributions to its completion. Their participation spanned from the implementation and design of the research to the analysis of the results and the writing of the manuscript.

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