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Fractional Order System Dynamical Behaviors with Beddington-DeAngelis Functional Response

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ABSTRACT

The present study proposes a fractional order prey-predator model with Beddington-DeAngelis functional response, that the Caputo fractional derivative is applied. There is exploration of the solutions' existence, uniqueness, non-negativity, and boundedness. Stability of all feasible equilibrium points is determined locally by the use of Matignon's condition. Moreover, the researchers also provide sufficient conditions to assure global asymptotic stability for both the predator-extinction equilibrium point and the positive equilibrium point, with selecting a relevant Lyapunov function and the incidence ofHopf-bifurcation is also displayed. Finally, the fractional order effect on the stability behavior of systems is investigated theoretically and also illustrated numerically to support theoretical results.

Keywords: Fractional calculus, Prey-predator model, Stability analysis, Hopf-bifurcation, Numerical simulation.

1. Introduction

Nowadays, the prey-predator systems dynamics is an interesting subject of study in both mathematical biology and ecology, and the stability of systems is regarded as one of the focus areas. The aim of this work is to explore a prey-predator model with Beddington-DeAngelis functional response[1-3]. For ecologists, it is a main goal to find out how prey and predator are related. One interesting aspect of relationship between prey-predator, is the predator's incidence of consuming prey[4-6] or so-called functional responses of predator. A number of notable functional response types have been identified such as: Holling types I–III[7, 8]; Beddington-DeAngelis type[9, 10]; Hassell–Varley type[11]; and the current famous ratio dependence type[12, 13]. Among these types, the Holling type I–III is marked "prey-dependent" and the types which contemplate the intrusion between predators are marked "predator-dependent"[12]. Currently, theoretical and experimental work in biology present much support to "predator-dependent" type models[4, 14-17]. However, no specific functional response fully explains all the sets of data. It is worthy to explore the Beddington-DeAngelis model as it can be created by several natural mechanisms[9, 16] and for the reason that it reveals rich but biologically practicable dynamics[18].

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Let prey and predator population densities be $X(t)$ and $Y(t)$ respectively, at any time t. The following is the form of Beddington-DeAngelis functional response of per capita consuming rate[9]:

$$R(X,Y)=\frac{aX}{1+bX+cY},$$

where a and b are positive constants, with units of (1/time) and (1/prey) respectively, which studies the impact of catch rate and processing time. How fast the consuming rate per capita proceeds toward its value of saturation a is dictated by b. cY (non-negative c units:1/predator) estimates the extent of reciprocal interaction among predators. Beddington-DeAngelis type and Holling type II functional responses are similar in that cY is in the denominator of both, which indicates the reciprocal interaction among predators.

Suppose that when the population of predator is absent, there is a strategic growth of prey population with $r$inherent rate of growth and K environmental carrying capacity, m denotes the highest value of reduction rate of $X(t)$ per capita because of $Y(t)$, a is attack rate and d is the rate of predator's death.The following is the proposed prey-predator model considered by Arditi and Ginzburg[12], which employs a Beddington-DeAngelis type functional response:

$$\frac{dX(t)}{dt} = rX(t)\left(1 - \frac{X(t)}{K}\right) - \frac{aX(t)Y(t)}{1 + bX(t) + cY(t)}\,,$$
$$\frac{dY(t)}{dt} = \frac{maX(t)Y(t)}{1 + bX(t) + cY(t)} - dY(t)\,, \tag{1}$$

$X(0) \geq 0$, $Y(0) \geq 0$.

For the sake of simplification, we nondimensionalize the system (1), using the followings:

$$x \longrightarrow \frac{X}{k}\ , \qquad y \longrightarrow aY\ , \qquad t \longrightarrow T\,,$$

So, the following is the transformation of the system (1):

$$\frac{dx(t)}{dt} = rx(t)\big(1 - x(t)\big) - \frac{x(t)y(t)}{1 + b_1x(t) + b_2y(t)}\,,$$
$$\frac{dy(t)}{dt} = \frac{b_0x(t)y(t)}{1 + b_1x(t) + b_2y(t)} - dy(t)\,, \tag{2}$$

where $b_1 = bk$, $b_2 = \frac{c}{a}$ and $b_0 = amk$.

All parameters are non-negative, for every time $t \geq 0$Due to its capacity to supply a solid explanation of definite nonlinear phenomena, differential equations with fractional order system have recently garnered interest of academics and researchers[19]. Ordinary differential equations can be generalized to arbitrary(non-integer) orders using fractional order differential equations. In diverse disciplines of engineering, chemistry and physics some scholars looked into differential equations with fractional order to explain complicated systems[20]. In recent years, numerous researchers have made use of biological models with fractional order[2, 21-28]. This is because differential equations with fractional order are inherently linked to memory systems[21]. Many systems of biology have memory, and the concept of system with fractional order might be more relevant to actual life conditions than that of system with integer order. The present study examines a prey-predator fractional order model with Beddington-DeAngelis functional response through expanding the integer order system (2), as following:

$${}^{c}D^{\alpha}x(t) = rx(t)\big(1 - x(t)\big) - \frac{x(t)y(t)}{1 + b_1x(t) + b_2y(t)}\,,$$
$${}^{c}D^{\alpha}y(t) = \frac{b_0x(t)y(t)}{1 + b_1x(t) + b_2y(t)} - dy(t)\,, \tag{3}$$

with the initial conditions, $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$, where $\alpha \in (0,1)$ and ${}^{c}D^{\alpha}$ is the Caputo fractional derivative.

So, the dynamics of a fractional order system (3) with functional response type of Beddington-DeAngelis is discussed, as well as the system's qualitative behavior. The stability of equilibrium points of system (3) is investigated on a local and global scale. In addition, the incidence of Hopf-bifurcation is demonstrated. Because of the region of stability for integer order system (2) is less than the region of stability for fractional order system (3), the dynamics of system (3) are more stable than those of the system (2)[29].

2. Preliminaries

We make extensive use of the Caputo fractional order derivative throughout this work, and this section will cover several definitions and practical lemmas.

Definition 1[30, 31] Caputo is a regularization of the Riemann Liouville formulation, as an alternative definition. For a function $x(t)$ with $t > t_0$, The α order of Caputo's derivative is characterized as

$${}^{c}D^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha+n-1}x^{(n)}(s)ds\,,$$

where $n - 1 < \alpha < n, n \in \mathbb{N}$ and ${}^{c}D^{\alpha}$ is the standard Caputo differentiation.

Following[32]. If $\alpha \in (0,1)$, then for a function $x(t)$ with $t > t_0$, Caputo's derivative of orderα reduced to

$${}^{c}D^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}x'(s)ds\,.$$

This paper primary focuses on Caputo's definition, that is more applicable in practice.

Lemma 1[33]. Let $\alpha \in (0,1]$. Assume that $x(t) \in C[h,k]$ and ${}^{c}D^{\alpha}x(t) \in C[h,k]$.

- **i.** If ${}^{c}D^{\alpha}x(t) \geq 0, for\ all\ t \in (h,k)$, then $x(t)$ is non-decreasing function for all $t \in [h,k]$.
- **ii.** If ${}^{c}D^{\alpha}x(t) \leq 0, for\ all\ t \in (h,k)$, then $x(t)$ is non-increasing function for all $t \in [h,k]$.

Definition 2[34] The equilibrium stability of an autonomous system involving the Caputo derivative ${}^{c}D^{\alpha} = Ax(t)$ with initial value $x(0) = x_0, \alpha \in (0,1)$, was explained and created by Matignon as follows:

- **i.** Stable iff for every x_0, there exists $\epsilon > 0$ such that $\| x(t) \| \leq \epsilon$ for $t \geq 0$;
- **ii.** Asymptotically stable iff $\lim_{t\to\infty} \| x(t) \| = 0$.

Global stability and globally asymptotically stable states are nearly identical to local stability and locally asymptotically stable states as defined in Definition 2.

Lemma 2[35] Let $x(t)$ be a derivable, continuous function on $\mathbb{R}_+$. Then, at any point in time, $t \geq t_0$

$${}^{c}D^{\alpha}\left[x(t) - \hat{k} - \hat{k}Ln\left(\frac{x(t)}{\hat{k}}\right)\right] \leq \left(1 - \frac{\hat{k}}{x(t)}\right)\ {}^{c}D^{\alpha}x(t), \quad for\ all\ \alpha \in (0,1),$$

where $\hat{k} \in \mathbb{R}_+$.

3. Analysis

3.1 Existence and uniqueness

Theorem 1. There is a unique solution of system (3), for each non-negative initial conditions.

Proof. We are looking for a necessary and adequate condition regarding existence and uniqueness of the solutions of system (3) in the domain $\Omega \times (0,T]$ where

$$\Omega = \{(x,y) \in \mathbb{R}^2 : max(|x|,|y|) \leq M\}.$$

Acquiring the approach used in[5]. Observe a mapping $G(X) = (G_1(U), G_2(U))$ and

$$G_1(U) = rx(1-x) - \frac{xy}{1+b_1x+b_2y}$$
$$G_2(U) = \frac{b_0xy}{1+b_1x+b_2y} - dy \qquad (4)$$

For any $U=(x,y), \tilde{U}=(\tilde{x},\tilde{y}),\quad U,\ \tilde{U}\in\Omega$, it comes from (4) that

$$\|G(U)-G(\tilde{U})\| = |G_1(U)-G_1(\tilde{U})| + |G_2(U)-G_2(\tilde{U})|$$

$$= \left|rx(1-x) - \frac{xy}{1+b_1x+b_2y} - r\tilde{x}(1-\tilde{x}) - \frac{\tilde{x}\tilde{y}}{1+b_1\tilde{x}+b_2\tilde{y}}\right| + \left|\frac{b_0xy}{1+b_1x+b_2y} - dy - \frac{b_0\tilde{x}\tilde{y}}{1+b_1\tilde{x}+b_2\tilde{y}} - d\tilde{y}\right|.$$

$$\le r|x-\tilde{x}| + r|x-\tilde{x}|(|x|+|\tilde{x}|) + d|y-\tilde{y}| + (1+b_0)$$
$$\frac{|y||x-\tilde{x}| + |\tilde{x}||y-\tilde{y}| + b_1|x||\tilde{x}||y-\tilde{y}| + b_2|y||\tilde{y}||x-\tilde{x}|}{F\ \tilde{F}},$$

where $F = 1+b_1x+b_2y$ and $\tilde{F} = 1+b_1\tilde{x}+b_2\tilde{y}$

$$\le [r(1+2M) + (1+b_0)M(1+Mb_2)]|x-\tilde{x}| + [d+(1+b_0)M(1+Mb_1)]|y-\tilde{y}|.$$

$$\le H_m\|U-\tilde{U}\|,$$

where $H_m = max\{(r(1+2M)+(1+b_0)M(1+Mb_2)), (d+(1+b_0)M(1+Mb_1))\}$.

So, $G(U)$ satisfy the condition of Lipschitz. As a consequence, the existence and uniqueness of system (3) are proven.

3.2 Non-negativity and boundedness

The densities of the interplaying populations are represented by system (3) solutions, that have to be non-negative and bounded. And these are guaranteed by the outcome below.

Theorem 2. All fractional order system (3) solutions are uniformly bounded and non-negative when beginning with $\Omega_+ = \{(x,y)\in\mathbb{R}^2 : x\ge 0, y\ge 0\}$.

Proof. All system (3) solutions beginning in Ω_+ are shown to be uniformly bounded.

Defining the function

$$W(t) = x(t) + \frac{1}{b_0}y(t).$$

Hence, for each $d>0$,

$$^cD^\alpha W(t) + dW(t) = {}^cD^\alpha\left(x(t)+\frac{1}{b_0}y(t)\right) + d\left(x(t)+\frac{1}{b_0}y(t)\right) = rx - rx^2 + dx.$$

Let $Q(x) = rx - rx^2 + dx$, by using second derivative test for a unique critical value $x_* = \frac{r+d}{2r}$, $Q''(x_*) = -2r < 0$, then the function $Q(x)$ has a maximum value at $x_* = \frac{r+d}{2r}$.

So, $Q(x) \le Q\left(\frac{r+d}{2r}\right) = \frac{(r+d)^2}{4r}\quad for\ all\ x\in[0,1]$.

Hence

$$^cD^\alpha W(t) + dW(t) \le \frac{(r+d)^2}{4r}$$

Using the standard fractional-order comparison theorem[36],

$$W(t) \le W(0)E_\alpha(-d\,t^\alpha) + \left(\frac{(r+d)^2}{4r}\right)t^\alpha E_{\alpha,\alpha+1}(-d\,t^\alpha),$$

where E_α is the Mittag-Leffler function, regarding[36],

$$W(t) \le \frac{(r+d)^2}{4r}\left(\frac{1}{d}\right) = \frac{(r+d)^2}{4rd},\quad \text{where}\quad t\to\infty.$$

All the system(3) solutions, beginning with Ω_+ are restricted to the domain W_1, where

$$W_1 = \left\{(x,y)\in\mathbb{R}_+^2 : W \le \frac{(r+d)^2}{4rd} + \epsilon, \epsilon > 0\right\}, \qquad (5)$$

After that, non-negativity of system (3)'s solutions is demonstrated.

Taking the first equation of system(3)

$$^cD^\alpha x(t) = rx(1-x) - \frac{xy}{1+b_1x+b_2y}. \qquad (6)$$

From (5) , we obtain this observation

$$W(t) = x + \frac{1}{b_0}y \le \frac{(r+d)^2}{4rd} = \theta_1. \qquad (7)$$

Depending on (6) *and* (7) we get

$$^cD^\alpha x(t) \ge rx(1-\theta_1) - b_0\theta_1 x = (r - r\theta_1 - b_0\theta_1)x = h_1x,\quad \text{where}\ h_1 = r - r\theta_1 - b_0\theta_1.$$

Regardingthe standard comparison theorem for fractional order[36]

$$x(t) \ge x(0)E_\alpha(h_1t^\alpha) + (0)t^\alpha E_{\alpha,\alpha+1}(h_1t^\alpha) = x_0E_\alpha(h_1t^\alpha).$$

Since $x(0) = x_0 \ge 0$ and by positivity of Mittag-Leffler function[37] ,then $E_\alpha(h_1t^\alpha) > 0$ for any $\alpha\in(0,1)$, then $x(t)\ge 0$.

From second equation of system (3)

$$^cD^\alpha y(t) = \frac{b_0xy}{1+b_1x+b_2y} - dy \ \ge -dy.$$

Regardingthe standard comparison theorem for fractional order[36]

$$y(t) \ge y(0)E_\alpha(-d\,t^\alpha) + (0)t^\alpha E_{\alpha,\alpha+1}(-d\,t^\alpha) = y_0E_\alpha(-d\,t^\alpha).$$

Since $y(0) = y_0 \geq 0$ and by Mittag-Leffler function's positivity[37], then $E_\alpha(-d\,t^\alpha) > 0$ for each $\alpha \in (0,1)$, then $y(t) \geq 0$.

Hence, the non-negativity of the system (3) solutions is proved.

3.3 Stability of equilibria

In order to determine the equilibrium points of system (3), we set ${}^cD^\alpha x(t) = 0$ and ${}^cD^\alpha y(t) = 0$, which are as below

- **i.** $E_0 = (0,0)$ is the trivial equilibrium point, which exists at all times.
- **ii.** $E_1 = (1,0)$ is the predator-extinction equilibrium point, which exists at all times.
- **iii.** $E_2 = (x_1, y_1)$ is the positive equilibrium point. According to the system (3) we get

$$r(1-x_1) - \frac{y_1}{1+b_1x_1+b_2y_1} = 0\,, \tag{8}$$

and

$$\frac{b_0x_1}{1+b_1x_1+b_2y_1} - d = 0\,. \tag{9}$$

From (8) and (9) we obtain

$$x_1 = \frac{1}{2rb_0b_2}\Big(-(b_0 - rb_0b_2 - db_1) + \sqrt{(b_0 - rb_0b_2 - db_1)^2 + 4db_0b_2}\Big)\,, \tag{10}$$

$$y_1 = \frac{1}{b_2}\left(\left(\frac{b_0 - db_1}{d}\right)x_1 - 1\right), \tag{11}$$

exists under the condition

$$0 < \frac{d}{b_0 - db_1} < x_1 < 1\,. \tag{12}$$

Hence, $E_2 = (x_1, y_1)$ is the only positive equilibrium point of system(3) which exists under the condition (12).
The Jacobian matrix is used to determine each equilibrium point stability.

From fractional order system(3), we set

$${}^cD^\alpha x(t) = rx(1-x) - \frac{xy}{1+b_1x+b_2y} = xP_1(x,y)\,, \tag{13}$$

$${}^cD^\alpha y(t) = \frac{b_0xy}{1+b_1x+b_2y} - dy = yP_2(x,y)\,,$$

where

$$P_1(x,y) = r(1-x) - \frac{y}{1+b_1x+b_2y} \text{ and } P_2(x,y) = \frac{b_0x}{1+b_1x+b_2y} - d\,.$$

The Jacobian matrix of (13), is obtained as follow

$$J(x,y) = \begin{bmatrix} x\dfrac{\partial P_1(x,y)}{\partial x} + P_1(x,y) & x\dfrac{\partial P_1(x,y)}{\partial y} \\ y\dfrac{\partial P_2(x,y)}{\partial x} & y\dfrac{\partial P_2(x,y)}{\partial y} + P_2(x,y) \end{bmatrix}, \tag{14}$$

where

$$\frac{\partial P_1(x,y)}{\partial x} = -r + \frac{b_1y}{F^2}\,, \quad \frac{\partial P_1(x,y)}{\partial y} = -\frac{1+b_1x}{F^2}$$

$$\frac{\partial P_2(x,y)}{\partial x} = \frac{b_0(1+b_2y)}{F^2}\,, \quad \frac{\partial P_2(x,y)}{\partial y} = -\frac{b_0b_2x}{F^2}$$

$F = 1 + b_1x + b_2y$ as stated previously.

The local stability of equilibrium points $E_0 = (0,0)$, $E_1 = (1,0)$, and $E_2 = (x_1, y_1)$ are being explored.

Theorem 3. The trivial equilibrium pointof system (3) $E_0 = (0,0)$ is a saddle point.

Proof. If either of the eigenvalues λ_i of the Jacobian $J(E_0)$ satisfy $|arg\ (\lambda_i)| < \frac{\alpha\pi}{2}$, then $E_0 = (0,0)$ is an unstable saddle point. In agreement with Matignon's condition[38, 39].

At $E_0 = (0,0)$, the Jacobian matrixofsystem (3) is as follows

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & -d \end{pmatrix}$$

For the characteristic equation of $J(E_0)$, these are the eigenvalues $\lambda_1 = r$, $\lambda_2 = -d$. clearly, we can see that $|arg\ (\lambda_1)| = 0 < \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1)$ and $|arg\ (\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1)$.

Theorem 4. Thepredator-extinction equilibrium point of system (3) $E_1 = (1,0)$, is locally asymptotically stable when

$$d > \frac{b_0}{1+b_1}\,, \tag{15}$$

and a saddle point when

$$d < \frac{b_0}{1+b_1}\,. \tag{16}$$

Proof. When every eigenvalues λ_i of the Jacobian $J(E_1)$ satisfy $|arg\ (\lambda_i)| > \frac{\alpha\pi}{2}$ by the Matignon's condition[38, 39], then $E_1 = (1,0)$ is locally asymptotically stable.

At $E_1 = (1,0)$, the Jacobian matrix of system (3)assessed as following

$$J(E_1) = \begin{pmatrix} -r & -\dfrac{1}{1+b_1} \\ 0 & \dfrac{b_0}{1+b_1} - d \end{pmatrix}.$$

For the characteristic equation of $J(E_1)$, these are the eigenvalues

$$\lambda_1 = -r \qquad \text{and}\ \lambda_2 = \frac{b_0}{1+b_1} - d$$

we can notice that

$|arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1)$, and if $d > \frac{b_0}{1+b_1}$, then $|arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1)$.

It indicates that under the condition (15), $E_1 = (1,0)$ is locally asymptotically stable. However,

if $d < \frac{b_0}{1+b_1}$, then $|arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1)$. and $|arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1)$.
Hence $E_1 = (1,0)$ is unstable saddle point under the condition (16).

According to Theorem 3 and Theorem 4 the stability of trivial and predator-extinction equilibrium point is not depending on the fractional derivative order α.

Now, the positive equilibrium point stability is explored. And at the positive equilibrium point$E_2 = (x_1, y_1)$, the Jacobian matrix of system (3) is also evaluated as in the following

$$J(E_2) = \begin{pmatrix} \frac{-rx_1{F_1}^2 + b_1x_1y_1}{{F_1}^2} & -\frac{x_1(1+b_1x_1)}{{F_1}^2} \\ \frac{b_0y_1(1+b_2y_1)}{{F_1}^2} & \frac{-b_0b_2x_1y_1}{{F_1}^2} \end{pmatrix},$$

where$F_1 = 1 + b_1x_1 + b_2y_1$.

The eigenvalues correlating with $E_2 = (x_1, y_1)$ are the following characteristic equation's roots$|J(E_2) - \lambda I| = 0$which becomes

$$\lambda^2 + T\lambda + D = 0\,, \qquad (17)$$

where

$$T = \frac{x_1}{{F_1}^2}\left(rx_1{F_1}^2 + (b_0b_2 - b_1)y_1\right) \text{ and}$$

$$D = \frac{b_0x_1y_1}{{F_1}^3}(rb_2x_1F_1 + 1) > 0\,.$$

We get the stability condition specified in the following theorem by studying the system's related characteristic equation at $E_2 = (x_1, y_1)$.

Theorem 5. The positive equilibrium point $E_2 = (x_1, y_1)$ of fractional order system (3) is

i. Locally asymptotically stable for all $\alpha \in (0,1)$, when $T > 0$.

ii. Unstable for all $\alpha \in (0,1)$, when$T < 0$ and $4D < T^2$.

iii. Locally asymptotically stable, when $T < 0$, $4D > T^2$ and

$$\tan^{-1}\left(\frac{-\sqrt{4D - T^2}}{T}\right) > \frac{\alpha\pi}{2}. \qquad (18)$$

iv. Unstable, when $T < 0$, $4D > T^2$ and

$$\tan^{-1}\left(\frac{-\sqrt{4D - T^2}}{T}\right) < \frac{\alpha\pi}{2}. \qquad (19)$$

Proof.

i. Following[40], two roots with negative real parts are present in the characteristic equation (17) when $T > 0$. Thus, under criterion of Routh-Hurwitz, the positive equilibrium point $E_2 = (x_1, y_1)$ for all $\alpha \in (0,1)$ is stable locally asymptotically.

ii. Following[40], the characteristic equation (17) has two eigenvalues as follows:

$$\lambda_{1,2} = \frac{-T \mp \sqrt{T^2 - 4D}}{2},$$

when $T < 0$ and $4D < T^2$. Then obviously $\frac{-T+\sqrt{T^2-4D}}{2} > 0$. Therefore, at least one positive real root present in the characteristic equation (17) . As a result, $E_2 = (x_1, y_1)$ is unstable for all $\alpha \in (0,1)$.

iii. Following[40], the characteristic equation(17) possesses pairs of complex conjugate eigenvalues with positive real parts as follows

$$\lambda_{1,2} = \frac{-T \mp \sqrt{T^2 - 4D}}{2} = \frac{-T \mp i\sqrt{4D - T^2}}{2}, \qquad (20)$$

is straight to obtain that

$$|arg(\lambda_{1,2})| = \tan^{-1}\left(\frac{-\sqrt{4D - T^2}}{T}\right), \qquad (21)$$

when $T < 0$, $4D > T^2$.

Following the Matignon's condition[38, 39], from (18) and (21) we get that the condition of stability is

$$\tan^{-1}\left(\frac{-\sqrt{4D - T^2}}{T}\right) > \frac{\alpha\pi}{2}.$$

If we set

$$\alpha = \frac{2}{\pi}\tan^{-1}\left(\frac{-\sqrt{4D - T^2}}{T}\right), \qquad (22)$$

then, for the increasing function (22) the maximum value of α occurs at

$$\alpha_M = \frac{2}{\pi}\tan^{-1}\left(\frac{-\sqrt{4D - T^2}}{T}\right), \qquad (23)$$

for $\alpha < \alpha_M$

$$\alpha < \alpha_M = \frac{2}{\pi}\tan^{-1}\left(\frac{-\sqrt{4D - T^2}}{T}\right) = \frac{2}{\pi}|arg(\lambda_{1,2})|, \quad \text{then } |arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}.$$

Following the Matignon's condition[38, 39], we get $E_2 = (x_1, y_1)$ is stable locally asymptotically for all $\alpha < \alpha_M$. That means, $E_2 = (x_1, y_1)$ is stable locally asymptotically for all $\alpha \in (0, \alpha_M)$.

for $\alpha > \alpha_M$

$$\alpha > \alpha_M = \frac{2}{\pi}\tan^{-1}\left(\frac{-\sqrt{4D - T^2}}{T}\right) = \frac{2}{\pi}|arg(\lambda_{1,2})|, \quad \text{then} |arg(\lambda_{1,2})| < \frac{\alpha\pi}{2}.$$

Following[39], we get $E_2 = (x_1, y_1)$ is unstable for all $\alpha > \alpha_M$.

So, the system (3) is stable locally asymptotically iff $T < 0 \; and \; 4D > T^2 \;\; for \; all \; \alpha \in (0, \alpha_M)$, and at all $\alpha > \alpha_M$ the system(3) is unstable.

iv. When $T < 0$, $4D > T^2$ and $\tan^{-1}\left(\frac{-\sqrt{4D-T^2}}{T}\right) < \frac{\alpha\pi}{2}$ then $|arg(\lambda_{1,2})| < \frac{\alpha\pi}{2}$.

Following[39], $E_2 = (x_1, y_1)$ is unstable.

From the third result of Theorem 5we can see, even though the eigenvalues of Jacobian matrix $J(E_2)$ has positive real part, the positive equilibrium point $E_2 = (x_1, y_1)$ of system(3) could be stable asymptotically.

3.5 Global stability

Here the stability of $E_1 = (1,0)$ and $E_2 = (x_1, y_1)$ globally asymptotically are investigated.

Theorem 6. When predator-extinction equilibrium point$E_1 = (1,0)$ of system (3) is stable locally asymptotically, it is also stable globally asymptotically.

Proof. Observe the positive defined Lyapunov function $V_1(x, y)$ interpreted as

$$V_1(x, y) = \frac{1}{1 + b_1 + b_2}\left(x(t) - 1 - Ln(x(t))\right) + \frac{1}{b_0}y(t)\,.$$

Alongside the solution of system(3), we compute the α order derivative of $V_1(x,y)$, it follows fromLemma 2that one has

$$\begin{aligned}{}^cD^\alpha V_1(x,y) &\le \frac{1}{1+b_1+b_2}\left(1-\frac{1}{x(t)}\right)\ {}^cD^\alpha x(t) \\ &+\frac{1}{b_0}\ {}^cD^\alpha y(t) \\ &\le -\frac{r}{1+b_1+b_2}(x-1)^2 \\ &+y\left(\frac{1+(b_1+b_2)x}{(1+b_1+b_2)(1+b_1x)}-\frac{d}{b_0}\right).\end{aligned}$$

Now, for increasing function $T(x)=\frac{1+(b_1+b_2)x}{(1+b_1+b_2)(1+b_1x)}$,$T(x)\le T(1)$ for all $x\le 1$, then $T(x)\le\frac{1}{1+b_1}$,and

$${}^cD^\alpha V_1(x,y)\le -\frac{r}{1+b_1+b_2}(x-1)^2+y\left(\frac{1}{1+b_1}-\frac{d}{b_0}\right).$$

Thus ${}^cD^\alpha V_1(x,y)\le 0$, where $\frac{1}{1+b_1} < \frac{d}{b_0}$. which is equivalent to the condition(15).

Regarding[41] it is followed that, under the condition (15), $E_1=(1,0)$ is stable globally asymptotically.

Note: The predator-extinction equilibrium point $E_1=(1,0)$ is stable locally and globally asymptotically under condition (15), which contradicts the condition (13). While condition (13) is an existence condition of y_1 in the positive equilibrium point $E_2=(x_1,y_1)$.

Hence, if the predator-extinction equilibrium point $E_1=(1,0)$ is locally or globally asymptotically stable, then the positive equilibrium point $E_2=(x_1,y_1)$ does not exist.

Theorem 7. Assume that

$$y_1<\frac{r}{b_1}, \tag{24}$$

then, the positive equilibrium point $E_2=(x_1,y_1)$ of system(3) is stable globally asymptotically.

Proof. Observe the positive defined Lyapunov function $V_2(x,y)$ interpreted as

$$\begin{aligned}V_2(x,y) &= \left(x(t)-x_1-x_1Ln\left(\frac{x(t)}{x_1}\right)\right) \\ &+N\left(y(t)-y_1-y_1Ln\left(\frac{y(t)}{y_1}\right)\right),\end{aligned}$$

where $N=\frac{1+b_1x_1}{b_0(1+b_2y_1)}$.

We compute the α order derivative of $V_2(x,y)$ alongside the solution of system (3), as it comes from Lemma 2, thus

$$\begin{aligned}{}^cD^\alpha V_2(x,y) &\le \left(1-\frac{x_1}{x(t)}\right)\ {}^cD^\alpha x(t) \\ &+N\left(1-\frac{y_1}{y(t)}\right)\ {}^cD^\alpha y(t), \\ &\le (x-x_1)\left(r-rx-\frac{y}{F}\right)+N(y-y_1)\left(\frac{b_0x}{F}-d\right).\end{aligned}$$

From the equation (8)and(9) we get that $r=rx_1+\frac{y_1}{F_1}$ *and* $d=\frac{b_0x_1}{F_1}$ respectively.

$$\begin{aligned}{}^cD^\alpha V_2(x,y) &\le (x-x_1)\left(rx_1+\frac{y_1}{F_1}-rx-\frac{y}{F}\right) \\ &+N(y-y_1)\left(\frac{b_0x}{F}-\frac{b_0x_1}{F_1}\right) \\ &\le -r(x-x_1)^2 \\ &-\frac{1}{FF_1}(-b_1y_1(x-x_1)^2+Nb_0b_2x_1(y-y_1)^2 \\ &+[(1+b_1x_1)-Nb_0(1+b_2y_1)](x-x_1)(y \\ &-y_1)).\end{aligned}$$

If we substitute the value of N ,we acquire

$$\begin{aligned}{}^cD^\alpha V_2(x,y) &\le -(x-x_1)^2(r-b_1y_1) \\ &-\frac{b_2x_1(1+b_1x_1)}{(1+b_2y_1)FF_1}(y-y_1)^2.\end{aligned}$$

In consequence, ${}^cD^\alpha V_2(x,y)\le 0$, *when* $y_1<\frac{r}{b_1}$.

Regarding[41], we get that if $y_1<\frac{r}{b_1}$, then the positive equilibrium point $E_2=(x_1,y_1)$ is stable globally asymptotically.

3.5 Hopf-bifurcation

A Hopf-bifurcation happens when an equilibrium point's stability alters and a limit cycle occurs at the same time that a parameter is altered[42] . The development of purely imaginary eigenvalues of the Jacobian matrix signifies the presence of Hopf-bifurcation in a system of integer order. When the bifurcation parameter is changed, the sign of the real component of the complex eigen values will change, thus the equilibrium point's stability behaviors change as well. This bifurcation also happens in a fractional order system when the order of the fractional derivative (α) is changed[43-45].

Theorem 8[44, 45] With applying the following conditions, if bifurcation parameter exceeds the critical value $\alpha_*\in(0,1)$, then the fractional order system (3) undergoes a Hopf-bifurcation at the equilibrium point

i. the Jacobian matrix $J(E_2)$has a pair of complex conjugate eigenvalues $\lambda_{1,2}=\varphi\mp i\beta$ *where* $\varphi>0$.
ii. $H(\alpha_*)=0$,where $H(\alpha)=\frac{\alpha\pi}{2}-\min_{1\le i\le 2}|arg(\lambda_i)|$.
iii. $\left.\frac{dH(\alpha)}{d\alpha}\right|_{\alpha=\alpha_*}\neq 0$.

Condition (ii) establishes that is the stability switching point, and condition (iii) ensures that $H(\alpha)$ can switch sign once the bifurcation parameter crosses the critical value α_*[46].

Then, when the derivatives order exceeds a critical value, we demonstrate the presence of a Hopf-bifurcation conditions at the positive equilibrium point$E_2=(x_1,y_1)$. The following theorem expresses the outcome.

Theorem 9[45] In system (3), if the bifurcation parameter α crosses the critical value $\alpha_M=\frac{2}{\pi}\tan^{-1}\left(\frac{-\sqrt{4D-T^2}}{T}\right)$, a Hopf-bifurcation arises at the positive equilibrium point $E_2=(x_1,y_1)$ if $T<0$ and $4D>T^2$.

Proof. Let $T<0$, $4D>T^2$ and the critical value $\alpha_M=\frac{2}{\pi}\tan^{-1}\left(\frac{-\sqrt{4D-T^2}}{T}\right)$. Define $\varphi=-\frac{1}{2}T$ and $\beta=\frac{1}{2}\sqrt{4D-T^2}$. By the first supposition, we obtain $\varphi=-\frac{1}{2}T>0$. Regarding the second supposition and equation (20), we possess a pair of complex conjugate eigenvalues $\lambda_{1,2} = \varphi \mp i\beta$, with $\varphi > 0$. So, condition (i) in Theorem 8 holds.

From both suppositions and $H(\alpha) = \frac{\alpha\pi}{2} - \min_{1\le i\le 2} |arg(\lambda_i)|$,we get

$$H(\alpha_M) = \frac{\alpha_M \pi}{2} - \min_{1\le i\le 2} |arg(\lambda_i)| = \frac{\alpha_M \pi}{2} - \tan^{-1}\frac{\beta}{\varphi}$$

$$= \tan^{-1}\left(\frac{-\sqrt{4D - T^2}}{T}\right) - \tan^{-1}\frac{\beta}{\varphi}$$

$$= \tan^{-1}\frac{\beta}{\varphi} - \tan^{-1}\frac{\beta}{\varphi} = 0.$$

Then condition (ii) in Theorem 8 holds.

And, based on the definition $H(\alpha)$, we obtain

$$\left.\frac{dH(\alpha)}{d\alpha}\right|_{\alpha=\alpha_M} = \frac{\pi}{2} \neq 0.$$

This indicates that condition (iii) is holds.

Hence, from Theorem 8, when bifurcation parameter α crosses the critical valueα_M, the system (3) undergoes Hopf-bifurcation at positive equilibrium point $E_2 = (x_1, y_1)$.

4. Numerical simulation

With the use MATLAB and following[47], this part includes numerical simulations to corroborate the theoretical conclusions discussed in the preceding portion of this paper. Additionally, the fractional order's influence on stability of equilibrium points is demonstrated using Beddington-DeAngelis functional response and the model of prey-predator. The majority of non-linear differential equations with fractional order lack accurate theoretical solutions, and numerical approaches are usually required to solve them. From biological point of view, these numerical simulations are very significant.

There are three sets of hypothetical values of parameters which is given in 3 cases as follows

Case1. $r = 0.53$,$b\_0 = 0.5$,$b\_1 = 1.5$,$b\_2 = 1.5$,$d = 0.24$

Case2. $r = 0.53$,$b\_0 = 0.4$,$b\_1 = 0.1$,$b\_2 = 0.6$,$d = 0.1$

Case3. $r = 0.53$,$b\_0 = 0.8$,$b\_1 = 0.5$,$b\_2 = 0.0012$,$d = 0.0021$

Regarding case1, with this collection of parameter values,

$$d = 0.24 > \frac{b_0}{1 + b_1} = 0.2 ,$$

which is the condition of stability both locally and globally asymptotically of $E_1 = (1,0)$for all $\alpha \in (0,1)$ which correspond with the result ofTheorem 4 and Theorem 6. The simulation result represents the time series of system(3), which starts at $x_0 = 0.4$, $y_0 = 1$ with $\alpha = 1$ and the phase plane of system (3) starts with various values of the initial conditions with $\alpha = 1$,the solution approaches asymptotically to the predator-extinction equilibrium point$E_1 = (1,0)$, with applying 1000 unit of time as seen in Figure 1(a and b).In Figure 2(a) the system(3)'s time series starts at $x_0 = 0.4$, $y_0 = 1$ with $\alpha = 0.9$ and the solution approaches$E_1 = (1,0)$ asymptotically. And in Figure 2 (b) the phase plane of system(3) starts with various values of the initial conditions with $\alpha = 0.9$ and the solution approaches$E_1 = (1,0)$ asymptotically, with increasing time to about 5000 unit.

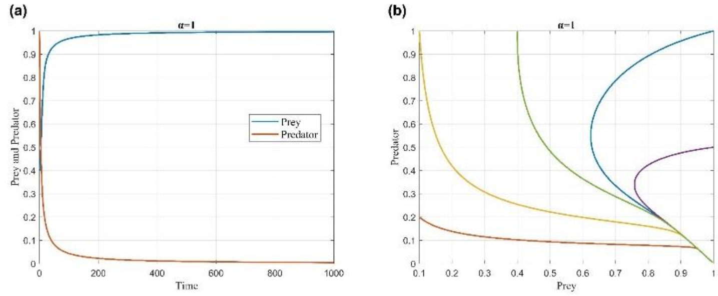

Figure 1: The fractional order system(3) with $\alpha = 1$, (a) The time series starts at $x_0 = 0.4$, $y_0 = 1$, (b) The phase plane starts with various values of the initial conditions.

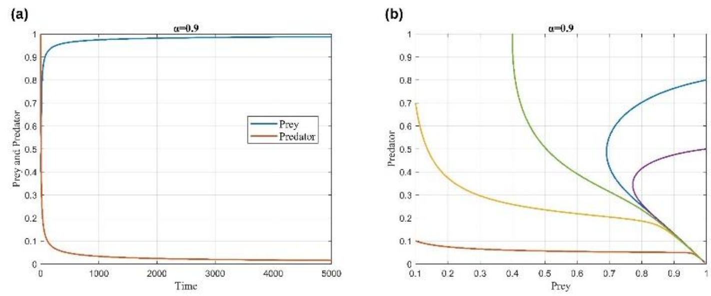

Figure 2: The fractional order system(3) with $\alpha = 0.9$, (a) The time series starts at $x_0 = 0.4$, $y_0 = 1$, (b) The phase plane starts with various values of the initial conditions.

Observing (Figure 1 and Figure 2),as proved theoretically that$E_1 = (1,0)$ is stable locally and globally asymptotically for all $\alpha \in (0,1)$, and its stability property is not depending on fractional derivative order α, from Theorem 4 and Theorem 6. We conclude that with decreasing the fractional derivative order α, the solution convergence speed will be slower, and a longer time will be needed for the solution to approach to the predator-extinction equilibrium point$E_1 = (1,0)$.

In Figure 3, Concerning thesolution of system (3) with parameter values in case1 for different α values, all numerical solutions are converging to$E_1 = (1,0)$, while with larger fractional derivative order α, solution of system (3) has higher convergence speed than with smaller α, and thesolution of system (3) approaches the integer-order system (2).

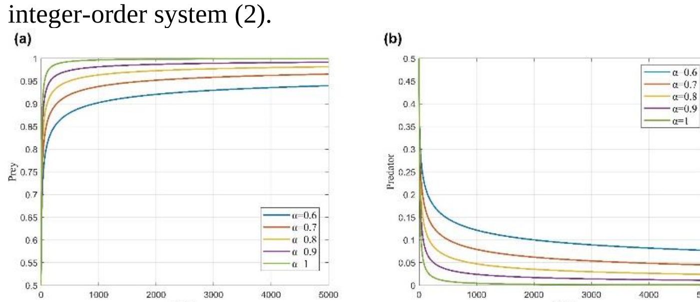

Figure 3: The solutions of the fractional order system(3) starts at $x_0 = 0.5$, $y_0 = 0.5$, for various values of α. (a) Effect of α on convergence speed of prey densities, (b) Effect of α on convergence speed of predator densities.

Considering case2 with the following parameter values that satisfy the condition

$$d = 0.1 < \frac{b_0}{1 + b_1} = 0.363$$

which also satisfy $D = 0.03 > 0$ and $T = 0.186 > 0$, $y_1 = 0.4682 < \frac{r}{b_1} = 5.3$ assures that the positive equilibrium point $E_2 = (0.3278{,}0.4682)$ is locally and globally asymptotically stable for all $\alpha \in (0,1)$, as shown theoretically in first result of Theorem 5 And also illustrated in (Figure 4 and Figure 5) in both time series and phase plane, with values of $\alpha = 1$ and $\alpha = 0.8$.

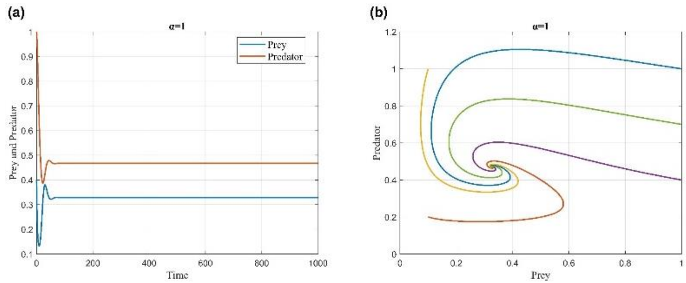

Figure 4: The fractional order system(3) with $\alpha = 1$, (a) The time series starts at $x_0 = 0.4$, $y_0 = 1$, (b) The phase plane starts with various values of the initial conditions.

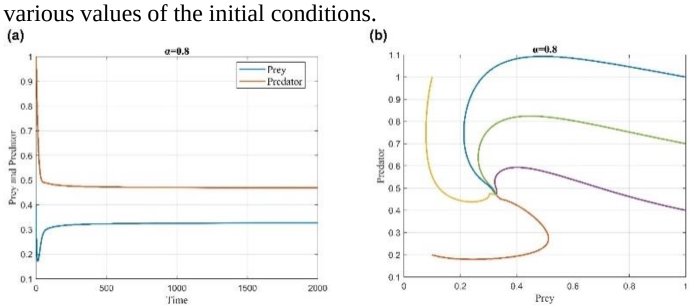

Figure 5: The fractional order system(3)with$\alpha = 0.8$, (a) The time series starts at $x_0 = 0.4$, $y_0 = 1$, (b) The phase plane starts with various values of the initial conditions.

Coming to the parameters of case3, they satisfy the conditions

$$d = 0.0021 < \frac{b_0}{1+b_1} = 0.533 \quad and \quad y_1 = 0.529 < \frac{r}{b_1} = 1.06$$

which also satisfy $T = -0.0025 < 0$, $4D = 0.0044 > T^2 = 0.00000625$

$$\text{and} tan^{-1}\left(\frac{-\sqrt{4D-T^2}}{T}\right) > \frac{\alpha\pi}{2} \qquad for\ all\ \alpha \in (0, \alpha_M)$$

where $\alpha_M = \frac{2}{\pi} tan^{-1}\left(\frac{-\sqrt{4D-T^2}}{T}\right) = 0.975434$.

The positive equilibrium point $E_2 = (0.00263{,}0.529)$ is stable locally and globally asymptotically for all $\alpha \in (0{,}0.975434)$, as proved theoretically in third result of Theorem5 and Theorem 7and also illustrated with both time series and phase plane in (Figure 6), with various value of α.

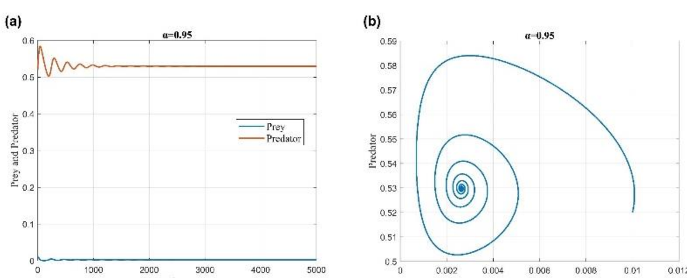

Figure 6: (a) The time series, and (b) The phase plane of fractional order system(3) starts at $x_0 = 0.01$, $y_0 = 0.52$with$\alpha = 0.95$.

Observe, for all $\alpha > 0.975434$ the positive equilibrium point $E_2 = (0.00263{,}0.529)$ start losing its stability and gradually undergoes a limit cycle behavior.as shown in Figure 7 and Figure 8

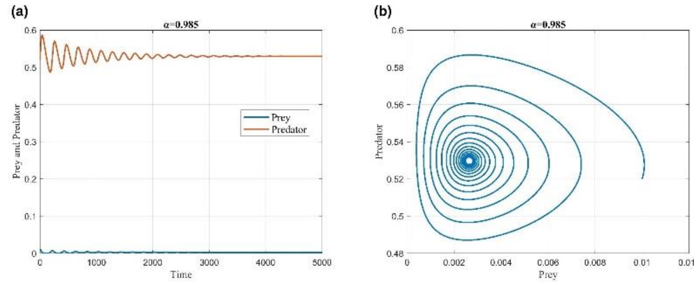

Figure 7: (a)The time series, and (b) The phase plane of fractional order system(3) starts at $x_0 = 0.01$, $y_0 = 0.52$with$\alpha = 0.985$.

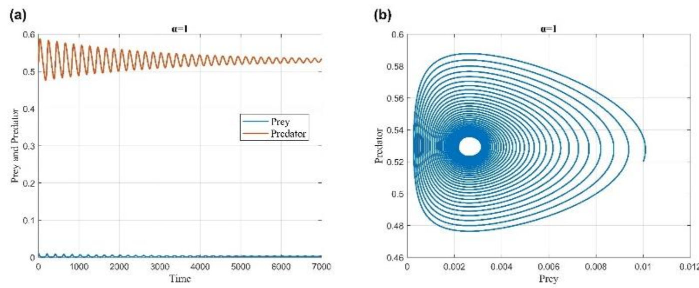

Figure 8: (a)The time series, and (b) The phase plane of fractional order system(3) starts at $x_0 = 0.01$, $y_0 = 0.52$with$\alpha = 1$.

Conclusion

The present article considers a fractional order prey-predator model using Beddington-DeAngelis functional response. the Beddington-DeAngelis functional response permits a variety of dynamics, such as the possibilities for extinction, persistence, stable or unstable equilibria, and limit cycles. Additionally, the fractional order system is an extension of the integer order system. So the researcher's aim is to represent both situations using a single model and to analyze it as there hasn't been much research on it and also it seems worthy of further study. First, the researchers demonstrate that the system has the properties of existence, uniqueness, non-negativity, and boundedness of solutions, which in population dynamics all are desirable. Then, the local stability of each equilibrium of system (3) is examined and by developing suitable Lyapunov functions, some sufficient condition for the global asymptotic stability of $E_1 = (1{,}0)$ and$E_2 = (x_1, y_1)$ is derived and occurrence of Hopf-bifurcation is exhibited. The stability of a positive equilibrium point might be influenced by the order of fractional derivative. The positive equilibrium point may be destabilized if the order α is greater than the critical order α_M . numerical verifications of analytical results are always required to complete analytical studies. As a result, MATLAB is used to numerically verify all our key analytical findings. Our numerical analysis reveals that system (3) has dynamical behavior at $E_1 = (1{,}0)$ and $E_2 = (x_1, y_1)$, which corroborates the analytical investigations. Furthermore, numerical studies reveal the impact of fractional order α and Beddington-DeAngelis functional responses on prey and predator population densities. Because of the fact that the fractional order system (3) has a bigger region of stability than that of the integer order system (2), the non-linear system (3) is more stable than the later.

Conflict of interests

None

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