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Some Nonstandard Treatment of the Singularity in the Differential Equation

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ABSTRACT

This paper aims to use some nonstandard concepts to find a nonstandard analytic and non-analytic infinitely close solution of the first-order ordinary differential equation in the monad of its singularity, where the differential coefficients are either infinitesimal, unlimited or have basic differential form. The obtained nonstandard solutions are more precise and compatible than the conventional ones. We named such a non-analytic infinitely close solution to the singularity by shadow solution. These cases of solutions are sometimes impossible to obtain by conventional methods.

KEYWORDS: Nonstandard Analysis, Ordinary Differential Equation, S-Continuity, Singularity, Monad, Infinitesimal.

1 INTRODUCTION

A. Robinson, who made a foundation of nonstandard analysis, was the first mathematician who formulates the principle of "infinitely small" in a logical manner by enlarging the set of real numbers to include infinitesimals and infinitely large (unlimited) quantities [1, 2]. Robinson's extension of real numbers provides a rigorous foundation for using infinitesimal and infinitely large quantities in analysis. There are several differential approaches to extending the formal analysis to the nonstandard. One of the famous approaches is the Internal Set Theory (IST) given by E. Nelson [3], which is assembled on the axiomatic set theory of ZFC. Any set illustrated in ZFC will be standard; moreover, we recognize that in the ZFC, any mathematical object: a function, an actual number… etc., will be a set. Also, every set, or we can say the formula in the IST, is said to be internal; sometimes it does not implicate the new predicate "standard" (that is, in case it will be a formula in the ZFC), else it said to be external [3, 4]. In this work, with the notation, we represent the appropriate extension of the traditional set of the real numbers R. Throughout this paper, the following definitions and theorems from the nonstandard analysis are considered:

Every element or set represented in a classical mathematics will be standard [5]. We say that x is a limited real number if $|x| \leq r$ for some positive standard real numbers r, or it is called unlimited whenever $|x| > r$ for all positive standard real numbers r, also it is called an infinitesimal whenever $|x| < r$ for all positive standard real numbers r, the set of every infinitesimal will be represented by $m(0)$. The only standard infinitesimal will be Zero [6]. If an infinitesimal x will be nonzero then $\frac{1}{x}$ will become unlimited.

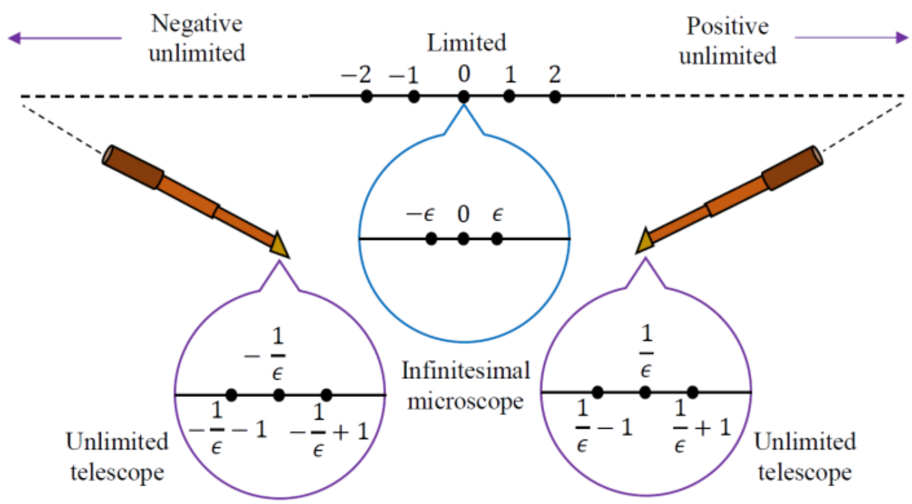

Figure 1: Structure of extended real numbers, where ϵ is an infinitesimal [7].

Let x and y be any two real numbers, then we say that x and y are infinitely close if and only if $x - y$ is infinitesimal. Also it will be denoted by the expression $x \cong y$ [8]. The set of every real number which is infinitely close to the real number x is said to be the monad of x also it will be represented by $m(x)$. A $\zeta -$ Microhal(x) $= \{y : y - x < \zeta^n, \forall^{st} n\}$, where ζ is an infinitesimal [1].

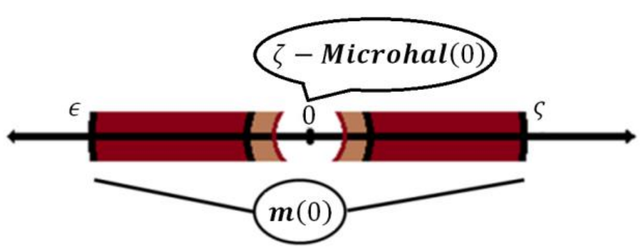

Figure 2: The $\zeta -$ microhal(0), where ζ, ϵ, and ς are infinitesimals.

For a limited number x in $\mathbb{R}$, will be infinitely close to a unique standard real number, and that unique number will say to be the shadow of x denoted by $st(x)$ or ^{o}x [3]. Any real number which is not infinitesimal and it is limited will be called appreciable [9]. For more details please see the references [10, 11]. A function $f: A \to B$ is said to be the internal function whenever A is an internal set, continuous at x_o if and only if f and x_o are standards and $f(x) \cong f(x_o)$ for all $x \cong x_o$, s-continuous if and only if $f(x) \cong f(x_o)$ for all $x \cong x_o$ [6, 9].

Let $(\mathbb{R}^n, d)$ be any standard metric space in $\mathbb{R}^n$. Then for any limited real numbers $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ for $n \in \mathbb{N}$, we define $m(x)$ to mean the monad of the point $(x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$, and x is infinitely close to y whenever $x_i \cong y_i$ for all $i = 1,2, \dots, n$. By $\mathrm{IntBnd}(x_1, x_2, \dots, x_n)$, we denoted to the interior of the bounded set and it is defined as $\mathrm{IntBnd}(x_1, x_2, \dots, x_n) = \{(y_1, y_2, \dots, y_n) \in X_n : |y_i| \leq x_i \text{ for all } i = 1,2, \dots, n\}$ [7]. Consider the following general form of first order differential equation:

$$\frac{dy}{dx} = f(x, y), \text{with initial condition } y(0) = 0. \tag{1}$$

Suppose that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are S-continuous functions defined on $\mathrm{IntBnd}(\gamma, \kappa)$, where γ and κ are positive real infinitesimals. Then there exists $x \in \gamma - \boldsymbol{Microhal}(0)$ so that $y = \psi(x)$ is a unique analytic solution to (1) [7]. For more details about nonstandard analysis see [4, 5, 6, 9, 12, 13].

Theorem 1[14]: The set X is a standard finite set if and only if every element of a set X is standard.

In the first order differential equation $M(x, y)dx + N(x, y)dy = 0$, if we have $M(x_0, y_0) = 0$ also $N(x_0, y_0) = 0$, then (x_0, y_0) is said to be a singular point for this differential equation [15]. A point x_0 is said to be a singular point whenever a function f is not analytic at this point while it is analytic at some point in every neighbourhood of x_0, and sometimes it's called a singularity of f. We mean that if a point $x = \vartheta$ will be a singular point to the solution $y = f(x)$ from the above DE then ϑ will become accumulation point for the set D that is $D = \{x \in \mathbb{C}: 0 < |x - \vartheta| < r, \text{for some } r \in \mathbb{R}^+\}$ wherever $y = f(x)$ will have a Taylor series about the point $x_0 \in D$ [16]. A function f in the open set Ω is said to be meromorphic while we have a sequence of points $\{x_0, x_1, x_2, \dots\}$ which does not have accumulation points in Ω, also this function f will be analytic in $\Omega - \{x_0, x_1, x_2, \dots\}$, moreover f has poles at $\{x_0, x_1, x_2, \dots\}$ [17]. A PDE is called quasilinear while it's linear w.r. to every highest order derivative for the unknown function.

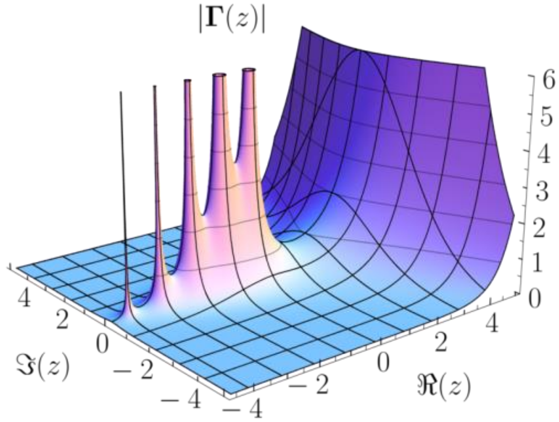

Figure 3: The gamma function is meromorphic in the complex plane [18].

In this investigation, we try to find the asymptotic behaviour of the solutions of the differential equation near singular points, which we call the shadow solution of the differential equation. Focusing on the properties of infinitesimal parameters related to such singular points in its monad leads to some unfamiliar behaviours and features of the singularity. Next, we give some nonstandard studies for ordinary differential equation (1) near a singularity.

Papers should clearly describe the background of the subject, the authors work, including the methods used, results and concluding discussion on the importance of the work. Papers are to be prepared in English and SI units must be used. Technical terms should be explained unless they may be considered to be known to the conference community. The references should be numbered [1], or [2, 3], or [1, 4-6].

2 NONSTANDARD SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATION

Consider the following general form of the first order ordinary differential equations

$$\frac{dy}{dx} = f(x, y), \tag{2}$$

where $x, y \in \mathbb{R}$. Existence and uniqueness theorems [7] assert that there exists a unique analytic solution $y_o = f(x_o)$ for $x = x_0$ when f(x, y) is analytic on $\mathbf{IntBnd}(|x_0| + \xi\ , |y_0| + \delta)$ where ξ and δ are positive real infinitesimals [9]. Next we discuss the analyticity of the solution of equation (2).

2.1. WHERE ONE OF THE DIFFERENTIAL COEFFICIENTS IS UNLIMITED

Theorem 2.1: Consider the differential equation (2) and assume that $f(x,y) = \frac{1}{P(x,y)}$, where $P(x,y)$ is a polynomial in x and y such that

$$P(x,y) = a_{1,0}x + a_{0,1}y + a_{1,1}xy + a_{2,0}x^2 + \cdots + a_{\omega_1,\omega_2}x^{\omega_1}y^{\omega_2},$$

where $a_{i,j}$ are limited real numbers for each $i,j \geq 0$ and ω_1 & ω_2 are unlimited real numbers with $\frac{1}{\omega_p} \notin \zeta - \mathbf{Microm}(0)$, where ζ is an infinitesimal, for $p = 1,2$. Then we have the following results:

1. If $(x_0, y_0) = (0,0)$, then the function $f(x,y)$ is not analytic in $\mathrm{m}(x_0, y_0)$. That is $f(x,y)$ is unlimited for all $(x,y) \in m(x_0, y_0)$.
2. $\frac{1}{f(x,y)}$ is analytic and $P(x,y)$ is an infinitesimal for all $(x,y) \in m(x_0, y_0)$.

Proof: Consider the differential equation

$$\frac{dx}{dy} = g(x,y) = a_0x + b_0y,$$

both functions $g(x,y)$ and $\frac{\partial g}{\partial x}(x,y)$ are defined for all y. The uniqueness theorem tells us that for each y_0 there exists a unique analytic solution x described in $\boldsymbol{m}(y_0)$. Now, let $z = a_0x + b_0y$. Then

$$\frac{dz}{dy} = a_0\frac{dx}{dy} + b_0 = a_0 z + b_0.$$

Hence, $\ln(a_0 z + b_0) = a_0 y + a_0c$, where c is an arbitrary constant. Therefore,

$$x(y) \cong \kappa + \gamma y + c_1 \sum_{n=2}^{\omega} \frac{(a_0y)^n}{n!},$$

where $c_1 = \frac{1}{{a_0}^2}e^{a_0c}$, $\kappa = c_1 - \frac{b_0}{{a_0}^2}$, $\gamma = c_1 a_0 - \frac{b_0}{a_0}$ and ω is unlimited with $\frac{1}{\omega} \notin \zeta - \mathbf{Microm}(0)$. For an infinitesimal y, we have $x(y) \cong \kappa$ where a_0 is appreciable, and if c_1 is infinitely close to $\frac{b_0}{{a_0}^2}$, then $x(0)$ becomes an infinitesimal. In general, if $g(x,y) = \sum_{i+j>0}^{\omega_1,\omega_2} a_{i,j}x^iy^j$ such that

$$\frac{dx}{dy} = \sum_{i+j>0}^{\omega_1,\omega_2} a_{i,j}x^iy^j. \quad (3)$$

Then, by the existence and uniqueness theorem, the approximate solution of the differential equation (3) will be of the form

$$x(y) = \sum_{i=1}^{\omega} c_iy^i, \quad (4)$$

which is zero for $y = 0$, and is infinitesimal for $y \in \zeta - \mathrm{Microhal}(0)$. Otherwise, the solution is nonzero. By using some interpolating method, we can find an infinitely close inverse y as a function of x such that

$$y(x) = \sum_{i=1}^{\omega} \hat{c}_i x^i. \qquad (5)$$

Thus equation (5) is a shadow solution for the differential equation (2) in $m(0)$ whenever $x \in \zeta -$ Microhal(0) and for standard infinitesimal $x = 0$, the differential equation (2) has no solution.■

2.2. WHERE THE DIFFERENTIAL COEFFICIENTS ARE EITHER INFINITESIMAL OR UNLIMITED

Theorem 2.2: Consider the following differential form of (2)

$$x\frac{dy}{dx} = f(x, y), \qquad (6)$$

and

$$f(x, y) = \alpha\, x\ + \beta\, y\ + \sum_{i+j>2}^{\omega_1,\omega_2} a_{i,j} x^i y^j\,,$$

where $a_{i,j}$ are limited real numbers for each $i, j \geq 0$ and ω_1 & ω_2 are unlimited real numbers with $\frac{1}{\omega_1}, \frac{1}{\omega_2} \in \left(\zeta - \text{Microhal}(0)\right)^c$, where ζ is an infinitesimal. More than this where $f(x, y)$ is analytic in $\boldsymbol{m}(0,0)$, is infinitesimal for $(x, y) \in \zeta - \text{Microhal}((0,0))$, with ζ is an infinitesimal, and $\frac{f(x,y)}{x}$ is meromorphic for (x, y) is infinitesimal in $\mathbb{C}^2$. Then we have the following cases:

- **I.** $\beta \in \mathbb{R}\backslash\{m(n): n \in \mathbb{N}^{st}\}$.
- **II.** $\beta \in \mathbb{R}\backslash\{m(n): n \in \mathbb{N}^{st}\}$ and α is an infinitesimal.
- **III.** α and β are infinitesimal.
- **IV.** $\beta \in \{m(n): n \in \mathbb{N}^{st}\}$.
  - **(a)** $\beta = 1$ and $\alpha = 0$.
  - **(b)** $\beta = 1$ and $\alpha \neq 0$.
  - **(c)** $\beta \in (m(1) \cap (\zeta - \boldsymbol{Microhal}(1)^c\}$.
  - **(d)** $\beta \in m(n)$ and n is limited natural number.
  - **(e)** β is unlimited and α is a limited.

Proof: By hypothesis, we can rewrite (6) as follows:

$$x\frac{dy}{dx} - \beta\, y\ = \alpha\, x + \textstyle\sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j} x^i y^j\ . \qquad (7)$$

It is clear that (0,0) is a singular point of the differential equation (7). First, we try to show that the differential equation (7) has a shadow solution in the monad of its singularity (0,0), then we try to show whether (7) has a solution or not for all cases given in the hypothesis and to determine the type and behavior of each existence solution. Let $\frac{dz}{dx} = g(x,z)$ be an equivalent differential equation to (7) such that $g(x,z)$ is meromorphic where (x,z) is infinitely close to the origin of $\mathbb{C}^2$. We can write these two differential equations in the following form:

$$\frac{dx}{1} = \frac{dy}{\frac{f(x,y)}{x}} = \frac{dz}{g(x,z)} .$$

The quasilinear partial differential equation of the above form is:

$$\frac{\partial y}{\partial x} + g(x,z)\frac{\partial y}{\partial z} = \frac{f(x,y)}{x}. \qquad (8)$$

We claim that the shadow solution of the equation (8) in the monad of its singularity is as follows:

$$y = \sum_{i+j\geq 1}^{\tilde{\omega}_1,\tilde{\omega}_2} \tilde{c}_{i,j} x^i z^j, \qquad (9)$$

where $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are unlimited, and assume that $g(x,y)$ is defined as:

$$g(x,y) = \frac{1}{x}\left(\hat{\alpha}\, x + \hat{\beta}\, y + \sum_{i+j\geq 2}^{\hat{\omega}_1,\hat{\omega}_2} \hat{a}_{i,j} x^i y^j\right).$$

Then (8) can be written as follows:

$$x\frac{\partial y}{\partial x} + \left(\hat{\alpha}\, x + \hat{\beta}\, z\right)\frac{\partial y}{\partial z} = \alpha\, x + \beta\, y + \sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j} x^i y^j - \left(\sum_{i+j>2}^{\hat{\omega}_1,\hat{\omega}_2} \hat{a}_{i,j} x^i y^j\right)\frac{\partial y}{\partial z}. \qquad (10)$$

Now, putting (9) and their derivatives in (10) yields

$$\hat{\beta}\tilde{c}_{0,1} = \beta\tilde{c}_{0,1}\ , \qquad \tilde{c}_{1,0} + \hat{\alpha}\tilde{c}_{0,1} = \alpha + \beta\,\tilde{c}_{1,0}.$$

Hence $\quad (\hat{\beta} - \beta)\tilde{c}_{0,1} = 0.$

Then either $\ \hat{\beta} = \beta\ $ or $\ \tilde{c}_{0,1} = 0.$

The coefficients $\left(\tilde{c}_{i,j}\right)_{i+j=k}$ must satisfy the following system

$$A_k \tilde{C}_k = F_k\left(\left(\tilde{c}_{i,j}\right)_{i+j<k}, a_{i,j}, \hat{a}_{i,j}\right) \text{, where}$$

$$A_k = \begin{pmatrix} k\hat{\beta} - \beta & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ k\hat{\alpha} & 1 + (k-1)\hat{\beta} - \beta & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (k-r+1)\hat{\alpha} & r + (k-r)\hat{\beta} - \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \hat{\alpha} & k - \beta \end{pmatrix},$$

$$\tilde{C}_k = \begin{pmatrix} \tilde{c}_{0,k} \\ \tilde{c}_{1,k-1} \\ \vdots \\ \tilde{c}_{r,k-r} \\ \vdots \\ \tilde{c}_{k,0} \end{pmatrix}, \text{ and } F_k = \begin{pmatrix} F_{0,k} \\ F_{1,k-1} \\ \vdots \\ F_{r,k-r} \\ \vdots \\ F_{k,0} \end{pmatrix}$$

For arbitrary nonzero $\tilde{c}_{0,1}$ the matrix A_k is invertible if and only if $\hat{\beta} = \beta$, where β is not a standard natural number and $\beta \notin \mathbb{Q}^-$ (where $\mathbb{Q}^-$ is the set of negative rational numbers), for all k $(k = 2,3,\cdots)$. Therefore, the differential equation (7) is equivalent to the following differential equation: $x\frac{dz}{dx} = \beta z$. The power series (9) is an approximate solution (shadow solution) of the following quasilinear partial differential equation

$$x\frac{\partial y}{\partial x} + \beta z\,\frac{\partial y}{\partial z} = \alpha x + \beta y + \sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}x^i y^j\,.$$

Hence, we get $y = \sum_{i+j\geq 1}^{\tilde{\omega}_1,\tilde{\omega}_2} \tilde{c}_{i,j}x^i\left(x^\beta\right)^j$,which is the approximate solution (shadow solution) of the differential equation (7) at the monad of its singularity. Consequently, if we choose β such $\beta \notin \mathbb{Q}^- \cup \mathbb{N}^{st}$with $\tilde{c}_{0,1} = 0$, then $\sum_{i=1}^{\tilde{\omega}_1} \tilde{c}_i x^i$ is the approximate solution (shadow solution) of the differential equation (7). Note that x^β is infinitesimal whenever x is infinitesimal. Now, if $\beta \in \mathbb{R}\backslash\mathbb{N}^{st}$, then the equation (7) has a unique analytical solution $\phi(x)$ such that $\phi(0) = 0$. In case $\phi(0) \cong 0$ then by Corollary 2.3 in [19] we obtain that equation (7) has an approximation solution (shadow solution). Let the power series about the origin for y be given as follows

$$y = \sum_{k=1}^{\omega} c_k x^k\,, \tag{11}$$

be an approximate solution (shadow solution) of the differential equation (7), where ω is unlimited such that $\frac{1}{\omega} \notin \zeta - \text{Microhal}(0)$. Now we try to find the radius of the coefficients c_i for real infinitesimal x. Assume that ρ is the maximum radius of coefficientsc_i, then we have the following cases:

1. If ρ is unlimited, then

$$|y| = \left|\sum_{k=1}^{\omega} c_k x^k\right| \leq \sum_{k=1}^{\omega} |c_k|\cdot\left|x^k\right| < \omega_0,$$

   where ω_0 is unlimited greater than ρ.

2. If ρ is appreciable, then

$$|y| = \left|\sum_{k=1}^{\omega} c_k x^k\right| \leq \sum_{k=1}^{\omega} |c_k|\cdot\left|x^k\right| \leq \sum_{k=1}^{\omega} \rho\cdot\left|x^k\right| = \frac{\rho}{1-|x|} \cong \rho.$$

3. If ρ is infinitesimal, then

$$|y| \leq \sum_{k=1}^{\omega} |c_k| \cdot |x^k| \leq \sum_{k=1}^{\omega} \rho \cdot |x^k| \lesssim 1 .$$

Now, we are ready to find the shadow solution of the equation (7) in the monad of its singularity according to given cases of α and β:

I - If $\beta \in \mathbb{R} \backslash \{\boldsymbol{m}(n): n \in \mathbb{N}^{st}\}$

Substituting (11) into (7) we obtain

$$x \sum_{k=1}^{\omega} k c_k x^{k-1} - \beta \sum_{k=1}^{\omega} c_k x^k = \alpha x + \sum_{i,j \geq 2}^{\omega_1, \omega_2} a_{i,j} x^i \left(\sum_{k=1}^{\omega} c_k x^k \right)^j .$$

That is

$$\sum_{k=1}^{\omega} (k - \beta) c_k x^k = \alpha x + \sum_{i,j \geq 2}^{\omega_1, \omega_2} a_{i,j} x^i \left(\sum_{k=1}^{\omega} c_k x^k \right)^j .$$

Therefore,

$$\begin{aligned}(1-\beta)c_1 x + (2-\beta)c_2 x^2 + \cdots + (\omega - \beta) c_\omega x^\omega \\ = \alpha x + \left(a_{2,0} + a_{1,1} c_1 + a_{0,2} {c_1}^2\right) x^2 \\ + \left(2a_{0,2} c_1 c_2 + a_{3,0} + a_{1,2} {c_1}^2 + a_{1,1} c_2 + a_{0,3} {c_1}^3 + a_{2,1} c_1\right) x^3 \\ + \cdots + P_\omega(c_1, c_2, \cdots, c_{\omega-1}) + \cdots \qquad (12)\end{aligned}$$

Thus,

$(1-\beta)c_1 = \alpha$

$(2-\beta)c_2 = a_{2,0} + a_{1,1} c_1 + a_{0,2} {c_1}^2$

$(3-\beta)c_3 = 2a_{0,2} c_1 c_2 + a_{3,0} + a_{1,2} {c_1}^2 + a_{1,1} c_2 + a_{0,3} {c_1}^3 + a_{2,1} c_1.$

In general, we have $(\omega - \beta) c_\omega = P_\omega(c_1, c_2, \cdots, c_{\omega-1})$, where P_ω is a polynomial in $c_1, c_2, \cdots, c_{\omega-1}$ with respect to the coefficients $a_{i,j}$ of the right-hand side of the equation (7). Let $\delta = \min\{|1-\beta|, |2-\beta|, \cdots, |\omega - \beta|\}$. This minimum exists for β is non-positive limited integer number because $|i - \beta| > 0$ for each $i = 1, 2, \cdots, \omega - 1$. Then we have the following cases for the minimum δ

a) If $\beta \cong 1$ with $\beta \neq 1$, then δ becomes an infinitesimal.

b) If $\beta \in \boldsymbol{m}(0)$, then the minimum number $\delta \cong 1$.

c) If β is unlimited, then δ also becomes unlimited except in the case that $\beta \in gal(\omega)$, that is the distance between ω and β is limited.

Now, recall equation (7) and let

$$F(x,y) = \sum_{i,j>0}^{\omega_1,\omega_2} a_{i,j} x^i y^j, \tag{13}$$

be a power series in two variables x and y, where x and y are complex or real numbers, and the coefficient $a_{i,j}$ is limited for each $i = 0,1,\cdots,\omega_1$ and $j = 0,1,\cdots,\omega_2$ where $\omega_1$and ω_2 are unlimited. Then for each $a_{i,j}$ there exist a limited $c_{i,j} \in \mathbb{R}^+$ such that $|a_{i,j}| \leq c_{i,j}$, for $i = 0,1,\cdots,\omega_1$ and $j = 0,1,\cdots,\omega_2$. Let $M = \max\{c_{i,j}: 0 \leq i \leq \omega_1 \text{ and } 0 \leq j \leq \omega_2\}$. Then

$$|f(x,y)| \leq \sum_{i,j=0}^{\omega_1,\omega_2} |a_{i,j}||x|^i|y|^j \leq \sum_{i,j=0}^{\omega_1,\omega_2} M|x|^i|y|^j \cong \frac{M}{(1-|x|)(1-|y|)},$$

for all $(x,y) \in \text{IntBnd}(1-\varsigma, 1-\varsigma)$, where ς is a positive real infinitesimal. Therefore, we have

$$\left|\alpha x + \sum_{i,j\geq 2}^{\omega_1,\,\omega_2} a_{i,j} x^i y^j\right| \leq \sum_{i,j=0}^{\omega_1,\,\omega_2} M|x|^i|y|^j - M(|x|)^0 - M(|y|)^1(|x|)^0,$$

Without restriction of generality we can assume that the variables are real numbers, then

$$x\frac{dy}{dx} - \beta y \leq \frac{M}{(1-x)(1-y)} - M - My\,.$$

Thus, $\frac{M}{(1-x)(1-y)} - M - My$ is an upper bound for equation (7) provided that α is limited. Consider the equation

$$\delta\,\hat{y} - H(x,\hat{y}) = 0, \tag{14}$$

where δ is minimum of $|n-\beta|$, $n \in \mathbb{N}$ and

$$H(x,\hat{y}) = \frac{M}{(1-x)(1-\hat{y})} - M - M\hat{y} = Ax + \sum_{i,j\geq 2}^{\omega_1,\,\omega_2} \hat{a}_{i,j} x^i \hat{y}^j\,,$$

for $\hat{y} = \sum_{k=1}^{\omega} \hat{c}_k x^k$ is an analytic series solution of equation (14), where ω is unlimited. Then

$$\delta \sum_{k=1}^{\omega} \hat{c}_k x^k - (Ax + \sum_{i,j\geq 2}^{\omega_1,\,\omega_2} \hat{a}_{i,j} x^i \left(\sum_{k=1}^{\omega} \hat{c}_k x^k\right)^j = 0.$$

That is, $\delta\,\hat{c}_1 x + \delta\,\hat{c}_2 x^2 + \cdots + \delta\,\hat{c}_\omega x^\omega = Ax + \left(\hat{a}_{2,0} + \hat{a}_{1,1}\hat{c}_1 + \hat{a}_{0,2}\hat{c}_1{}^2\right)x^2$

$$+\left(2\hat{a}_{0,2}\hat{c}_1\hat{c}_2 + \hat{a}_{3,0} + \hat{a}_{1,2}\hat{c}_1{}^2 + \hat{a}_{1,1}\hat{c}_2 + \hat{a}_{0,3}\hat{c}_1{}^3 + \hat{a}_{2,1}\hat{c}_1\right)x^3 + \cdots + P_\omega(\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_{\omega-1}) + \cdots.$$

Thus,

$\delta\,\hat{c}_1 = A$

$\delta\,\hat{c}_2 = \hat{a}_{2,0} + \hat{a}_{1,1}\hat{c}_1 + \hat{a}_{0,2}\hat{c}_1{}^2$

$\vdots$

$\delta\,\hat{c}_\omega = P_\omega(\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_{\omega-1}),$

where P_ω is a polynomial in $\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_{\omega-1}$ with the coefficients of the series $H(x, y)$. Since $0 < \delta \le |n - \beta|$, then $\hat{y}$ is an upper bound of y. Put $\Lambda = \max\{|\hat{c}_1|, |\hat{c}_2|, \cdots, |\hat{c}_\omega|\}$, then $\hat{y}$ become an infinitesimal whenever $x = \frac{x_0}{\Lambda}$ and since $\hat{y}$ is an upper bound of y, then also y becomes an infinitesimal. For an illustration consider the following differential equation form:

$$x\frac{dy}{dx} - \beta y = ax,$$

then $y = \frac{a}{1-\beta}x + c\ x^{-\beta}$ is general solution, where c is an arbitrary constant and according to the value of β we have the following solution results of y:

i. The solution y becomes unlimited in the following situations:

1. $\beta \in m(1)$ where a, x and c are appreciable.
2. Either β is positive unlimited and $|x| < 1$ or β is negative unlimited and $|x| > 1$, for limited a and c.
3. Both c and β are unlimited and a is limited with $x = 1$.
4. $\beta \in m(1) \cap (\zeta - \mathrm{Microm}(1))^c$ where ζ is a positive real infinitesimal and

4.1. If $a \in \zeta - \mathrm{Microm}(0)$, then $y \cong cx$. Therefore if c (or x) is unlimited and x (or c) is appreciable or both of them are unlimited, then y becomes unlimited.

4.2. If $x \in \zeta - \mathrm{Microm}(0)$, then $y \cong a$. Thus if a is unlimited then y is also unlimited.

ii. The solution y becomes infinitely close to the singularity (that is y becomes infinitesimal) in the following cases:

1. $\beta \notin m(1)$ and β, a and c are limited with $x \in m(0)$.
2. Either β is positive unlimited and $|x| > 1$ where x is limited or β is negative unlimited and $|x| < 1$, where both a and c are limited.
3. x and β are appreciable with $\beta \notin m(1)$ and c and a are infinitesimals.
4. $\beta = \hat{\alpha} + i\hat{\beta}$ where $\hat{\alpha}$ and $\hat{\beta}$ are limited real numbers. If we assume that r is the radius and θ is the argument of x, then we have $|x^\beta| = r^{\hat{\alpha}} e^{-\theta\hat{\beta}}$.

4.1 If $\hat{\alpha}$ is a positive infinitesimal and $r \in m(0)$, then y becomes an infinitesimal.

4.2 If $\hat{\alpha}$ is a negative infinitesimal or standard infinitesimal and $\hat{\beta} \notin \mathrm{IntBnd}(1)$ it suffices to assume that θ is a function of r such that $\hat{\alpha}\mathrm{Log}(r) - \hat{\beta}\theta(r)$ becomes negative unlimited as $r \in m(0)$ (for example take $\theta(r) = \hat{\alpha}(\mathrm{Log}(r))^2$) to make y infinitesimal.

Note that, for $a \cong 0$, the solution reduces to a linear solution. $y \cong c\,x^{-\beta}$, whose standard part is exactly ${}^{o}y = {}^{o}(c\,x^{-\beta})$, this leads to the result that the shadow solution for $a \cong 0$ is a linear asymptote for the general solution in $m(0)$. According to the cases discussed in the above illustration for $\beta \notin m(n)$, where n is limited natural number, and β is limited real number, then for $x \in \zeta - \text{Microhal}(0)$ the solution y becomes infinitely close to the singularity, that is y is infinitesimal shadow solution of singularity of the differential equation (7).

Let α and $\frac{1}{n-\beta}$, for each $n \in \mathbb{N}$, be limited real numbers. Since each $a_{i,j}$ is limited for $i = 0, 1, \cdots, \omega_1$ and $j = 0, 1, \cdots, \omega_2$ with $i + j \geq 2$, then the coefficients c_k of the equation (11) becomes limited for $k = 1, 2, \cdots, \omega$. Hence there exists positive standard reals $\tilde{c}_k \in \mathbb{R}$ such that

$$|c_k| \leq \tilde{c}_k \text{ for each } k = 1, 2, \cdots, \omega.$$

Put $\mathcal{M} = \max\{\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_\omega\}$, then $\sum_{k=1}^{\omega} \mathcal{M} x^k$ is an upper bound of (11) and it becomes an infinitesimal when $x = \frac{x_0}{\mathcal{M}}$ and $x_0 \in \zeta - \text{Microhal}(0)$.

II - If $\beta \in \mathbb{R}\backslash\{m(n): n \in \mathbb{N}^{st}\}$ and α is an Infinitesimal

Assume that $\beta \notin \boldsymbol{m}(n)$ where n is a limited natural number and β is limited, and let $\alpha, a_{i,0} \in \zeta - \boldsymbol{Microhal}(0)$, for $i = 2, \cdots, \omega_1$, then the equation (7) will become

$$x\frac{dy}{dx} - \beta y \cong \sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}\ x^i y^j.$$

Since $(1-\beta)c_1 \cong 0$ and $(2-\beta)c_2 \cong a_{2,0} + a_{1,1}c_1 + a_{0,2}c_1^2 \cong 0$ for β is limited, then $c_1 \cong 0$ and $c_2 \cong 0$. Hence the coefficients of (11) becomes infinitesimal. Now let $\tau = \max\{|c_1|, |c_2|, \cdots, |c_\omega|\}$, if τ is an infinitesimal, then in general we will have:

$$(\omega - \beta)c_\omega \cong P_\omega(c_1, c_2, \cdots, c_{\omega-1}).$$

Thus, $c_\omega \cong 0$, and

$$\sum_{i=k}^{\omega} |x| \geq \sum_{i=k}^{\omega} |x^k| >> \sum_{i=k}^{\omega} |c_k x^k| \geq |y|\,. \tag{15}$$

Therefore, the analytic solution y becomes infinitesimal either if $x \in \zeta - \text{Microhal}(0)$, or $x = \frac{x_0}{\tau}$, where $x_o \in \zeta - \text{Microhal}(0)$.

III - Where α and β are Infinitesimals

Assume that $\alpha, \beta, a_{i,0} \in \zeta - \text{Microhal}(0)$, for $i = 2, \cdots, \omega_1$. Then, the equation (7) will become

$$x\frac{dy}{dx} \cong \sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}\, x^i y^j.$$

Suppose that equation (11) is the shadow solution for the above differential equation. Then

$c_1 \cong 0,$

$2c_2 \cong a_{2,0} + a_{1,1}c_1 + a_{0,2}c_1^2 \cong 0,$

$\vdots$

In general,

$\omega c_\omega \cong P_\omega(c_1, c_2, \cdots, c_{\omega-1}),$

where P_ω is a polynomial in $c_1, c_2, \cdots, c_{\omega-1}$ with respect to the coefficients $a_{i,j}$. Now, let $\eta = \max\{|c_1|, |c_2|, \cdots, |c_\omega|\}$ and $x_0 \in \zeta - \text{Microhal}(0)$. We have either η is limited or unlimited. If η is limited then we take $x = x_0$ and for the other case, we choose the independent variable $x = \frac{x_0}{\eta}$. Hence, in both cases the solution (11) become the shadow solution of the singularity.

IV - If $\beta \in \{\mathbf{m}(n): n \in \mathbb{N}^{st}\}$

Recall equation (7)

$$x\frac{dy}{dx} - \beta y = \alpha x + \sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}\, x^i y^j.$$

Then we have the following results about the solution of the differential equation (7)

IV – a. $\beta = 1$ **and** $\alpha = 0$.

Assume that $\beta = 1$, then from (12) we obtain that $\alpha = 0$. Hence (7) becomes

$$x\frac{dy}{dx} - y = \sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}\, x^i y^j.$$

Now, put $y = xz$, then we get

$$x\left(x\frac{dz}{dx} + z\right) - xz = \sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}\, x^i (xz)^j.$$

Thus,

$$\frac{dz}{dx} = \frac{1}{x^2}\sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}\, x^i (xz)^j .$$

By existence and uniqueness Theorems the previous equation has an analytic solution at $m\big((0,0)\big)$. Assume that this solution given by

$$z = \sum_{i=1}^{\omega} k_i x^i .$$

Thus, equation (7) has unlimited number of analytical solutions at $m\big((0,0)\big)$ and zero at $x = 0$.

IV – b. $\beta = 1$ **and** $\alpha \neq 0$.

Without loss of generality, we consider the following form of the differential equation (7).

$$x\frac{dy}{dx} - y = \alpha x,$$

where α is non-zero constant. Then its solution is given by:

$$x^{-1}y = \alpha \int x^{-1}\, dx + c \text{ or } y = x(\alpha \operatorname{Log}(x) + c),$$

where c is an arbitrary constant. Thus, there is no analytical solution at $x = 0$. If x is infinitesimal, then x is in the extended complex domain, so let r be the radius and θ be the argument of x, with r being nonstandard positive infinitesimal and $\theta \cong 0$. To make $y \in m(0)$, it suffices that $\alpha \cong 0^+$ such that $\alpha \operatorname{Log}(r)$ is limited.

In general, if $\beta = 1$ and α is non-zero in the equation (7), then we obtain the solution depending on an arbitrary constant given from a power series in x , xLog(x) and they are infinitesimals for $x = r\, e^{i\theta}$ under the condition given for r and θ in the previous example. Now, assume that x is infinitesimal, $\beta = 1$ and $\alpha = 0$, then the differential equation (7) has unlimited analytical solutions. Moreover if x is standard infinitesimal, that is $x = 0$, then (7) has zero solution.

IV – c. $\beta \in \left(m(1) \cap \left(\zeta - \text{Microhal}(1)\right)^c\right)$.

Assume that and α is an infinitesimal such that $\alpha, a_{i,j} \in \zeta - \text{Microhal}(0)$, for $i = 2, \cdots, \omega_1$. Then from (12), we get that c_1 an infinitesimal. Since $c_2 = \frac{a_{2,0}+a_{1,1}c_1+a_{0,2}c_1^2}{2-\beta}$ therefore c_2 is also an infinitesimal. In general

$$(\omega - \beta)c_\omega = P_\omega(c_1, c_2, \cdots, c_{\omega-1}).$$

So, we get c_ω as an infinitesimal whenever the maximum of the radius of the coefficients of (11) is infinitesimal. Since $\beta \in m(1)$ and α is appreciable, then the coefficients c_i in (11) are unlimited and zero at $x = 0$. This gives us that (7) has y is an infinitesimal solution. Therefore, from (11), and for $x \in \zeta - \text{Microhal}(0)$ we obtain that y is an infinitesimal.

IV – d. $\beta \in m(n)$ **and** n **is a limited natural number**

We consider the transformation $y = \frac{a}{1-\beta}x + z\,x$, then

$$\frac{dy}{dx} = \frac{a}{1-\beta} + z + x\frac{dz}{dx}.$$

Put the last result in (7), we get

$$x\left(\frac{a}{1-\beta} + z\right) + x^2\frac{dz}{dx} - \beta x\left(\frac{a}{1-\beta} + z\right) = \alpha x + \sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}x^i\left(\frac{a}{1-\beta}x + zx\right)^j.$$

Divide both sides by x, we will get

$$x\frac{dz}{dx}-(\beta-1)z = \alpha_1 x+\phi_1(x,z). \tag{18}$$

By a suitable transformation if we make $\phi_1(x,z)$ through the origin. Then (18) becomes the same as (7), with $\beta-1$ instead of β. A finite number of transformations will reduce the problem to the case (3.4.1). In general, if β is an appreciable natural number and α is standard infinitesimal then (7) has an unlimited number of analytical solutions at $m\big((0,0)\big)$ and zero at $x=0$, and if α is not standard infinitesimal then there does not exist any analytical solutions at the origin for the equation (7).

IV – e. β **is unlimited and** α is **limited**

The coefficients c_i , for $i=1,2,\cdots,\omega$ of (11) become infinitesimals whenever $\frac{1}{1-\beta}\in\zeta-\boldsymbol{Microhal}(0)$ and $\eta, a_{i,0}\in\zeta-\mathrm{Microhal}(0)$, for $i=2,\cdots,\omega_1$. Hence, the solution y is infinitesimal for any limited $x\in\zeta-\mathrm{Microhal}(0)$. Thus, the solution becomes infinitely close to the singularity.

2.3 Irreducible Differential Form

Consider the first-order differential equation having irreducible differential form. That is the differential equation of the form:

$$M(x,y)dx+N(x,y)dy=0,$$

where $M(x,y)$ and $N(x,y)$ are power series in x and y. Therefore, assume that

$$(ax+by+\sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}x^iy^j)dx+(\hat{a}x+\hat{b}y+\sum_{i+j\geq 2}^{\omega_3,\omega_4}\hat{a}_{i,j}x^iy^j)\,dy=0. \tag{19}$$

So, the differential coefficients are convergent in the $m\big((0,0)\big)$ and $(0,0)$ is a singular point for (19), where ω_k for k=1, 2, 3, 4 is unlimited with $\prod_{k=1}^{4}\frac{1}{\omega_k}\notin\zeta-\mathrm{Microhal}(0)$ and the coefficients of the variables are limited. By the transformation $y=xz$ equation (19) becomes

$$(ax+bxz+\sum_{i+j\geq 2}^{\omega_1,\omega_2} a_{i,j}\,x^{i+j}\,z^j)dx\;+(\hat{a}x+\hat{b}xz+\sum_{i+j\geq 2}^{\omega_3,\omega_4}\hat{a}_{i,j}x^{i+j}z^j)(xdz+zdx)=0$$

So,

$$x\frac{dz}{dx}=-\frac{a+\big(\hat{a}+\hat{b}\big)z+\hat{b}z^2+\sum_{i+j\geq 2}^{\omega_5,\omega_6}\alpha_{i,j}\,x^i\,z^j}{\hat{a}+\hat{b}z+\sum_{i+j\geq 2}^{\omega_7,\omega_4}\hat{a}_{i,j}x^iz^j},$$

where $\omega_5 = (\max\{\omega_1, \omega_3\} + \max\{\omega_2, \omega_4\}) - 1$, $\omega_6 = \max\{\omega_2, \omega_4\}$ and $\omega_7 = (\omega_3 + \omega_4) - 1$. We know that the geometric series $1 + g(x,y) + \left(g(x,y)\right)^2 + \cdots$ converges to $\frac{1}{1-g(x,y)}$ whenever $|g(x,y)| < 1$ and it diverges when $|g(x,y)| \geq 1$. Thus,

$$1 + g(x,y) + \left(g(x,y)\right)^2 + \cdots + \left(g(x,y)\right)^{\omega} \cong \frac{1}{1-g(x,y)}, \tag{20}$$

where ω is unlimited and $\frac{1}{\omega} \notin \zeta -$ Microhal(0).

Case1: If a is appreciable, then the equation (20) becomes

$$\frac{dz}{dx} \cong \frac{1}{\frac{-x}{a}\left(\hat{a} + \hat{b}z \sum_{i+j\geq 2}^{\omega_7,\omega_4} \hat{a}_{i,j} x^i z^j\right)\left(1 - g(x,z) + \cdots + \left(-g(x,z)\right)^{\omega}\right)},$$

where $g(x,z) = \frac{1}{a}\left((\hat{a}+b)z + \hat{b}z^2 + \sum_{i+j\geq 2}^{\omega_5,\omega_6} \alpha_{i,j}\, x^i\, z^j\right)$ because $|g(x,z)| < 1$, whenever $(x,z) \in \zeta -$ Micrhalo $\left((0,0)\right)$. Hence, the problem is the same as the case in the section (2), where one of the differential coefficients is unlimited whenever x and z are standard infinitesimals.

Case2: If $\hat{a}$ is appreciable, and $a \in m(0)$, then the equation (20) will take the form:

$$x\frac{dz}{dx} \cong \frac{1}{\hat{a}}\left(((\hat{a}+b)z + \hat{b}z^2 + \sum_{i+j\geq 2}^{\omega_5,\omega_6} \alpha_{i,j} x^i z^j)(1 - g(x,z) + \left(g(x,z)\right)^2 - \cdots + (-1)^{\omega}\left(g(x,z)\right)^{\omega}\right)),$$

where $g(x,z) = \frac{1}{\hat{a}}(\hat{a} + \hat{b}z + \sum_{i+j\geq 2}^{\omega_7,\omega_4} \hat{a}_{i,j} x^i z^j$ because $|g(x,z)| < 1$, whenever $(x,z) \in \zeta -$ Microhal$((0,0))$. This problem is the same as the case in the section (3), where the differential coefficients are infinitesimals or unlimited whenever x and z are standard infinitesimal. The left-hand side of (20) is an infinitesimal whenever $x \in m(0)$, then we have

$$a + (b+\hat{a})z + \hat{b}z^2 \cong 0.$$

If x is standard infinitesimal, then

$$a + (b+\hat{a})z + \hat{b}z^2 = 0$$

ii. If $\hat{a} + \hat{b}z \notin m(0)$ then equation (20) reduces to the case as in Section 2.2, where the differential coefficients are both standard infinitesimals or unlimited and it has an analytical solution.

iii. If $\hat{a} + \hat{b}z \in \boldsymbol{m}(0)$ and the other coefficients are appreciable then (20) will have no analytical solution as x is infinitely close to the singularity.

3 DISCUSSION AND CONCLUSIONS

We have expressed the coefficient of the and as power series in two variables passing through the origin. If it is not at the centre, we can do a limited-distance translation to make it pass through. Then, we found a nonstandard analytic solution for the first-order differential equation in the monad of singularity, while sometimes in classical mathematics, there are no solutions for some differential equations with nonstandard analysis; by using external and internal sets, one can find a precise, perfect nonstandard solution for them.

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