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Finding Approximate Roots of Nonlinear Equations by Using New Sixth-Order Convergence

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ABSTRACT

In this study, we introduced new iterative techniques, for the purpose solving nonlinear equations. The new approaches are based on the auxiliary equation and Newton's method, respectively. Our method's a convergence analysis is explained. The novel approach is found to have convergence order of six. Numerical experiments show that the new method work better than well-known iterative methods and is similar to them.

KEYWORDS: Newton’s Raphson, Iterative Methods, Auxiliary Equation, Order of Convergence

1 INTRODUCTION

Numerical analysis is a branch of mathematics and computer science that develops, analyses, and solves continuous mathematics problems numerically. These problems come up when algebra, geometry, and calculus are used in the real world. They involve variables that change over time and can be found in the natural sciences, social sciences, engineering, medicine, and business, among other fields. A novel iterative strategy has been developed to find approximate solutions to the nonlinear equation $f(x) = 0$. For further information, read [1-11]. This numerical approach was created by using many different methods, such as Taylor series, homotopy [8], the quadrature formula [9], and the decomposition method [10].

In this paper, we propose a two- steps iterative method having sixth- order converge by using the Taylor series and the auxiliary equation defined in [1].

2 ITERATIVE METHODS

Suppose that the nonlinear equation is

$$f(x) = 0 \tag{1}$$

Let x_0 be the known initial guess for the necessary root and γ be a simple root. Assume

$$x_1 = x_0 + h, \qquad |h| \ll 1 \tag{2}$$

by using the Taylor series for $f(x_0 + h)$

$$f(x_0 + h) = f(x_0) + hf'(x_0) + O(h^2) \tag{3}$$

we are looking for a small h such as

$$f(x_0 + h) = 0 \approx f(x_0) + hf'(x_0) \tag{4}$$

giving

$$h = -\frac{f(x_0)}{f'(x_0)} \tag{5}$$

Consider the following auxiliary equation of (Noor and Ahmed, 2006) with a parameter p

$$g(x) = p^3(x - x_0)^2 f^2(x) - f(x) = 0 \tag{6}$$

where $p \in R$ and $|p| < \infty$.

It is clear, the root of equation (1) is also the root for equation (6) and vice versa. To the better approximate for the required root use $x_1 = x_0 + h$, then equation (6) gives

$$p^3 h^2 f^2(x_0 + h) - f(x_0 + h) = 0 \tag{7}$$

Expanding equation (7) by Taylor theorem, obtained

$$0 \approx -f(x_0) - hf'(x_0) + h^2\left(p^3 f^2(x_0) - \frac{f''(x_0)}{2}\right) \tag{8}$$

From equation (5) and equation (8), we get

$$h = -\frac{2f(x_0)f'(x_0)}{2f'^2(x_0) + 2p^3 f^2(x_0) - f''(x_0)f(x_0)} \tag{9}$$

The following one-step iterative approach can be used to solve the nonlinear equations $f(x) = 0$.

Algorithm 1: The value of p is selected such that the signs of $f(x_n)$ and p are the same. Calculate x_1, x_2, ... for a given x_0 iterative schemes,

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) + 2p^3 f^2(x_n) - f''(x_n)f(x_n)}$$

Know, by combine Algorithm 1 and Newton's method, we obtained the following two-step iterative method.

Algorithm 2: The value of p is selected such that the signs of $f(x_n)$ and p are the same. Calculate x_1, x_2, ... for a given x_0 iterative schemes,

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) + 2p^3f^2(x_n) - f''(x_n)f(x_n)}$$
$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}$$

3 CONVERGENCE ANALYSIS

The convergence analysis will be discussed for the iterative method in the following theorem, where we used Mathematica program 9 to prove that the convergence order for Algorithms 2 is six.

Theorem 1: A sufficiently differentiable function $f: I \in R \to R$ for an open interval I should have a simple zero $\gamma \in I$. The iterative strategy of Algorithm 2 demonstrates sixth-order convergence if x_0 is closely enough to γ.

Proof: The technique is given by

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) + 2p^3f^2(x_n) - f''(x_n)f(x_n)} \tag{10}$$
$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} \tag{11}$$

Let γ be a simple zero of f. By the Taylor expansions,

$$f(x_n) = f'(\gamma)\left(e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)\right) \tag{12}$$
$$f'(x_n) = f'(\gamma)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)\right) \tag{13}$$

and

$$f''(x_n) = f'(\gamma)\left(2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + O(e_n^4)\right) \tag{14}$$

where $c_k = \frac{f^{(k)}(\gamma)}{k!f'(\gamma)}, k = 2, 3, \ldots$ and $e_n = x_n - \gamma$. Multiple equations (12) by (13) and multiple equations (14) by (12), gives us

$$f(x_n)f'(x_n) = f'^2(\gamma)\left(2e_n + 6c_2e_n^2 + 2(2c_2^2 + 4c_3)e_n^3 + O(e_n^4)\right) \tag{15}$$
$$f(x_n)f''(x_n) = f'^2(\gamma)\left(2c_2e_n + (2c_2^2 + 6c_3)e_n^2 + (8c_2c_3 + 12c_4)e_n^3 + O(e_n^4)\right) \tag{16}$$

take the power for equations (12) and (13)

$$f^2(x_n) = f'^2(\gamma)\left(e_n^2 + 2c_2e_n^3 3 + O(e_n^4)\right) \tag{17}$$
$$f'^2(x_n) = f'^2(\gamma)\left(1 + 4c_2e_n + (4c_2^2 + 6c_3)e_n^2 + (12c_2c_3 + 8c_4)e_n^3 + O(e_n^4)\right) \tag{18}$$

After an elementary calculation, we obtain

$$
\begin{aligned}
&2f'^2(x_n)+2p^3f^2(x_n)-f(x_n)f''(x_n)\\
&\qquad = f'^2(\gamma)\Big(2+6c_2e_n+(-2c_2^2-6c_3+2(4c_2^2+6c_3)+2p^3)e_n^2\\
&\qquad + (-8c_2c_3-12c_4+2(12c_2c_3+8c_4)+4c_2p^3)e_n^3+O(e_n^4)\Big)
\end{aligned}
\tag{19}
$$

From equations (15) and (19), we obtain

$$
\frac{2f(x_n)f'(x_n)}{2f'^2(x_n)+2p^3f^2(x_n)-f(x_n)f''(x_n)} = e_n+(-c_2^2+c_3-p^3)e_n^3+O(e_n^4) \tag{20}
$$

Thus, from equations (10) and (20), we obtained

$$
\begin{aligned}
y_n &= \gamma+(c_2^2-c_3+p^3)e_n^3+(-3c_2^3+6c_2c_3-3c_4-c_2p^3)e_n^4\\
&\quad +(6c_2^4-18c_2^2c_3+6c_3^2+12c_2c_4-6c_5-p^6)e_n^5\\
&\quad +\Big(-9c_2^5-29c_2^2c_4+19c_3c_4-10c_6+3c_4p^3+c_2^3(37c_3+4p^3)\\
&\quad +c_2(-28c_3^2+20c_5-7c_3p^3+2p^6)\Big)e_n^6+O(e_n^7)
\end{aligned}
\tag{21}
$$

Also expanding $f(y_n)$ and $f'(y_n)$ about γ, we have

$$
\begin{aligned}
f(y_n) &= f'(\gamma)\big[(c_2^2-c_3+p^3)e_n^3+(-3c_2^3+6c_2c_3-3c_4-c_2p^3)e_n^4\\
&\quad +(6c_2^4-18c_2^2c_3+6c_3^2+12c_2c_4-6c_5-p^6)e_n^5\\
&\quad +\big(-9c_2^5-29c_2^2c_4+19c_3c_4-10c_6+3c_4p^3+c_2^3(37c_3+4p^3)\\
&\quad +c_2(-28c_3^2+20c_5-7c_3p^3+2p^6)+c_2(c_2^2-c_3+p^3)^2\big)e_n^6\\
&\quad +O(e_n^7)\big]
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
f'(y_n) &= f'(\gamma)\big[1+2c_2(c_2^2-c_3+p^3)e_n^3+2c_2(-3c_2^3+6c_2c_3-3c_4-c_2p^3)e_n^4\\
&\quad +2c_2(6c_2^4-18c_2^2c_3+6c_3^2+12c_2c_4-6c_5-p^6)e_n^5\\
&\quad +2c_2\big(-9c_2^5-29c_2^2c_4+19c_3c_4-10c_6+3c_4p^3+c_2^3(37c_3+4p^3)\\
&\quad +c_2(-28c_3^2+20c_5-7c_3p^3+2p^6)+(c_2+1.5c_3)(c_2^2-c_3+p^3)^2\big)e_n^6\\
&\quad +O(e_n^7)\big]
\end{aligned}
\tag{23}
$$

From (11), (21), (22) and (23) we have

$$
x_{n+1} = \gamma+\frac{1}{6}(c_2^3-c_2c_3+p^3c_2)^2e_n^6+O(e_n^7) \tag{24}
$$

This implies that

$$
e_{n+1} = \frac{1}{6}(c_2^3-c_2c_3+p^3c_2)^2e_n^6+O(e_n^7) \tag{25}
$$

This shows that the two-step iterative method given in Algorithm 2 has sixth-order convergences.

4 NUMERICAL EXAMPLE

In this article, we provide several numerical examples to demonstrate the effectiveness of the newly designed iterative methods. The methods of Muhammad and Faizan [1] (MF), Rostam and Shno [2] (RS), Manoj and Arvind [3] (MA), Shuping and Youhua [4] (SY), Obadah and Isha [5] (OI), Najmuddin and Vimal [6] (NV), Srinivasarao and Shanmugasundaram [7] (SS) and the method of Newton Raphson we compare with RMO (Algorithm 2), the scheme proposed in this work. As a result, for computer programs, the following stopping conditions are used:

i. $|x_n - x_{n-1}| < \epsilon$

ii. $|f(x_n)| < \epsilon$

We used $\epsilon = 10^{-15}$ and taking the following examples in [1-6]

$f_1(x) = \sin^2(x) - x^2 + 1$

$f_2(x) = (x-1)^3 - 1$

$f_3(x) = x^3 + 4x - 10$

$f_4(x) = x^2 - (1-x)^5$

$f_5(x) = e^{x^2+7x-30} - 1$

$f_6(x) = x^2 - e^x - 3x + 2$

$f_7(x) = \cos(x) - x$

Tables 1, also show the number of iterations required to approach the zero (IT), the approximate zero x_n, the value of $|f(x_n)|$ and $|x_n - x_{n-1}|$.

Table 1 Examples and comparisons between different methods.

	IT	$\boldsymbol{x_n}$	$\boldsymbol{|f(x_n)|}$	$\boldsymbol{|x_n - x_{n-1}|}$
$f_1, x_0 = 1$				
NM	6	1.404491	3.331e-16	3.060e-13
MF	3	1.404491	3.331e-16	6.247e-07
RS	3	1.404491	3.331e-16	1.147e-04
OI	3	1.404491	6.53e-114	1.67e-19
SY	2	1.404491	4.441e-16	5.208e-04
MA	2	1.404491	3.331e-16	1.018e-03
NV	5	1.404491	3.331e-16	1.028e-09
SS	2	1.404491	4.441e-16	4.485e-06
RMO	2	1.404491	2.44e-16	7.92e-12

$f_2, x_0 = 2,5$				
NM	6	2	0	1.155e-14
MF	3	2	0	1.067e-07
RS	3	2	0	8.882e-16
OI	3	2	7.91e-127	9.62e-22
SY	2	2	0	1.678e-03
MA	2	2	0	8.432e-04
NV	5	2	0	4.393e-09
SS	2	2	0	2.157e-03
RMO	2	2	0	2.05e-12
$f_3, x_0 = 1$				
NM	5	1.365230	0	2.127e-11
MF	3	1.365230	0	2.127e-11
RS	3	1.365230	0	3.255e-03
OI	3	1.556773	0	1.573e-11
SY	2	1.365230	0	8.102e-06
MA	2	1.365230	0	6.725e-06
NV	5	1.365230	0	7.834e-13
SS	2	1.365230	0	4.441e-04
RMO	2	1.556773	0	1.31e-12
$f_4, x_0 = 0.3$				
NM	4	0.345954	6.939e-17	6.431e-11
MF	2	0. 345954	6.939e-17	7.581e-06
RS	3	0.345954	6.939e-17	9.520e-08
OI	3	0.345954	3.62e-18	4.832e-21
SY	2	0.345954	6.939e-17	3.638e-08
MA	2	0.345954	6.939e-17	1.004e-07
NV	4	0.345954	1.943e-16	4.061e-12
SS	2	0.345954	6.939e-17	3.933e-10
RMO	1	0.345954	6.94e-17	8.53e-11
$f_5, x_0 = 3,5$				
NM	12	3	0	2.531e-13
NA	6	3	0	3.34e-42

RS	6	3	0	3.567e-13
OI	4	3	0	8.182e-06
SY	5	3	0	8.453e-05
MA	-	NaN*	-	-
NV	-	NaN	-	-
SS	5	3	0	2.352e-10
RMO	3	3	0	1.00e-04
$f_6, x_0 = 2$				
NM	5	0.257530	0	9.864e-14
MF	3	0.257530	1.0e-3	7.6e-21
RS	3	0.257530	0	9.876e-07
OI	3	0.257530	2.85e-130	8.73e-22
SY	4	0.257530	0	3.955e-05
MA	3	0.257530	0	2.799e-11
NV	5	0.257530	0	1.929e-13
SS	2	0.257530	0	1.819e-05
RMO	2	0.257530	0	1.913e-07
$f_7, x_0 = 1{,}7$				
NM	54	0.739085	4.441e-16	3.259e-08
MF	3	0.739085	0	1.22e-35
RS	2	0.739085	0	1.025e-03
OI	3	0.739085	6.67e-207	1.68e-34
SY	2	0.739085	0	2.329e-04
MA	2	0. 739085	1.110e-16	4.060e-04
NV	4	0.739085	0	1.088e-08
SS	2	0.739085	0	1.795e-04
RMO	2	0.739085	0	1.35e-10

* NaN means we obtain 0/0 for the formula.

5 CONCLUSION

Table 1 shows that the efficiency of our techniquewe found in Algorithm 2 obtain results with lower or the same number of iterations, for each of the (3rd order, 4th order, 6th order, and 7th order) schemes or Newton's method itself.

We recommendation this method extended to the system of nonlinear or adding another method to algorithm 2 for creating a new three steps iterative method.

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