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Original Article

Solvability for Couple Elliptic Partial Differential Equations

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ABSTRACT

As it famous, the solvability of any mathematical problem play an essential role by many investigators whose interest in the numerical solution for such mathematical problems. Because it gives these investigators the green light about the ability to solve such problems numerically. Various types of real life issues are described mathematically either by ODEs or by PDEs. The solvability for these types of problems (ODEs and PDEs) are studied usually in finite dimensional space, and are studied in infinite dimensional space (Hilbert space) rarely. In the recent years, the interesting for studying more general type of PDEs (system of PDEs) began arise, especially in infinite dimensional spaces. For these reasons, this work is devoted to study the solvability (in infinite dimension) of couple elliptic partial differential equations (CEPDEs) with four different types of boundary conditions (BCs): Dirichlet (DBCs), Neumann(NBCs), Robin(RBCs) and Mixed BCs(MBCs). The weak formulation (WF) for each problem according to the type of each BC is found. The existence and the uniqueness of the "couple" solution for the issues proposed (the CEPDEs with every kind of BCs) is proved by employing the Lax-Milgram theorem (LMTH) by using the Poincare-Friedrichs inequality (PFI) in the cases of homogenous BCs (HBCs) and through using the generalization PFI (GPFI) and the trace Operator (TRO) in the cases of nonhomogeneous BCs(NHBCs). The results were obtained in this work give the green light to the researchers about seeking the numerical solution for these types of problems.

Keywords: Lax-Milgram Theorem, Weak Formulation, Couple Elliptic Partial Differential Equations, Dirichlet Conditions, Neumann Condition, Robin Condition, Mixed Boundary Conditions.

1 Introduction

Many real-life problems in various and different kinds of sciences like fluid physics [1], vibrations of solids [2], quantum mechanics [3], the flow of fluids [4], the spread of heat [5], the structure of molecules [6], the interactions of photons and electrons [7], and the radiation of electromagnetic waves [8], the description of perturbations of the pressure and radius in arterial blood flow [9] are described as mathematical models that are usually represented by ODEs or PDEs.

In addition, differential equations also play an essential role in many field of sciences like as; modern mathematics and their analysis [10], qualitative theory [11], finance [12], optimal control theory [13] and recently, machine learning (ML) [14]. Hence, the numerical solution for PDEs became a fundamental task for many investigators to solve different complex mathematical problems [15] and complex physical phenomena like wave propagation [16], diffusion [17], heat transfer [18], and the interaction for electromagnetic radiation and mechanical waves with the Earth's surface [19], besides to compute the mechanical wave modeling, electromagnetic [20]. The study of solvability PDEs represents a cornerstone to the study of numerical solutions of these PDEs. It plays a significant role in formulating and implementing numerical methods, and it ensures the convergence to the solution. Also, the solvability has a direct influence on the stability of the solution resulting from a numerical method, and it avoids unreliable solutions form emphasizing the necessary of well-posed problems.

As a result, for the above studies, this paper is devoted to study the solvability (in infinite dimension) of the proposed CEPDEs based on LMTH with different types of boundary conditions (BCs), Dirichlet (DBCs), Neumann(NBCs), Robin(RBCs) and Mixed BCs(MBCs). The WF of the proposed CEPDEs with each of the four types of BCs was found at first. Then secondly, the existence and uniqueness of the solution to each WF are demonstrated by using the LMTH [21] under suitable assumptions with the help of the (PFI) in the case of homogenous BCs (HBCs) for each type and through using the generalization PFI (GPFI) and the trace operator (TRO) in the case of nonhomogeneous

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BCs(NHBCs) for each type. This study is very important since it has a significant influence on ensuring the solvability for interesting and complex mathematical issues in [22-27], besides the other natural problems.

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As a result, for the above studies, this paper is devoted to study the solvability (in infinite dimension) of the proposed CEPDEs based on LMTH with different types of boundary conditions (BCs), Dirichlet (DBCs), Neumann(NBCs), Robin(RBCs) and Mixed BCs(MBCs). The WF of the proposed CEPDEs with each of the four types of BCs was found at first. Then secondly, the existence and uniqueness of the solution to each WF are demonstrated by using the LMTH [21] under suitable assumptions with the help of the (PFI) in the case of homogenous BCs (HBCs) for each type and through using the generalization PFI (GPFI) and the trace operator (TRO) in the case of nonhomogeneous BCs(NHBCs) for each type. This study is very important since it has a significant influence on ensuring the solvability for interesting and complex mathematical issues in [22-27], besides the other natural problems.

2 Basic Concepts

This section includes some definitions, theorems and lemmas that interest in our work.

Definition2.1 [28]**:** Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \le p \le \infty$ the Sobolev Space $W^{1,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ whose distributional first-order partial derivatives belong to $L^p(\Omega)$; that is for all i=1,2,…,N there exists a function $g_i \in L^p(\Omega)$ such that:

$$\int_\Omega u \frac{\partial \theta}{\partial x_i}\, dx = -\int_\Omega g_i \theta \, dx \text{ , for all } \theta \in c_c^\infty(\Omega).$$

The function $g_i$is called the weak partial derivatives of u to x_i , and is denoted $\frac{\partial u}{\partial x_i}$.

Theorem 2.2 (LMTH) [21]**:** Assume V is a Hilbert space, $a(.,.)$ is bounded and V-elliptic bilinear form on V, $l \in V^*$. Then, there is a unique solution to the problem.

$$u \in V,\ a(u, v) = l(v), \quad \forall\, v \in V .$$

Lemma 2.3(PFI) [29]**:** For any Lipschitz bounded domain Ω, there is a constant $c(\Omega) > 0$, which depends only on the domain Ω such that $\int_\Omega |y|^2\, dx \le \int_\Omega |\nabla y|^2\, dx,\ \forall y \in H_0^1(\Omega)$.

Lemma2.4 (GPFI) [29]**: L**et $\Omega \subset \mathbb{R}^N$ be Lipschitz bounded domain, and let $\Gamma_1 \subset \Gamma$ be a measurable set such that $|\Gamma_1| > 0$. Then there exist a constant $c(\Gamma_1) > 0$, which is independent of $y \in H^1(\Omega)$, such that $\|y\|^2_{H^1(\Omega)} \le c(\Gamma_1)(\int_\Omega |\nabla y|^2\, dx + \left(\int_\Gamma y\, ds\right)^2)$, for all $y \in H^1(\Omega)$.

Theorem 2.5(TRO) [30]**:** Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz bounded domain, then there exists a linear operator (TRO) $T_0: H^1(\Omega) \to L^2(\Gamma)$, s.t.

$$T_0 u = u_{|\Gamma} \text{ if } u \in C^\infty(\bar{\Omega}), \text{ and } \|T_0 u\|_{L^2(\Gamma)} \le c_r \|u\|_{H^1(\Omega)}$$

For a suitable constant $c_r = c_r(\Omega, d)$; $T_0 u$ is called the trace of u on Γ.

3 Description of the Proposed Problem

Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz- bounded domain with boundary $\Gamma = \partial\Omega$; consider, the following CEPDEs with HDBCs:

$$A_1 y_1 + B_1 y_1 + c_1(x) y_1 - d(x) y_2 = f_1(x) , \tag{1}$$

$$A_2 y_2 + B_2 y_2 + c_2(x) y_2 + d(x) y_1 = f_2(x), \tag{2}$$

$$y_1 = 0 \quad on \ \ \Gamma, \tag{3}$$

$$y_2 = 0 \quad on \ \ \Gamma, \tag{4}$$

where $A_l y_l = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{lij}(x) \frac{\partial y_l}{\partial x_j}\right)$, $B_l y_l = \sum_{i=1}^{2} b_{li}(x) \frac{\partial y_l}{\partial x_i}$, $l = 1,2, \vec{y} = (y_1, y_2)$, $X = (x_1, x_2) \in \Omega$ $\vec{y} \in \left(H_0^2(\Omega)\right)^2$, $f_l(x) \in L^2(\Omega)$, and a_{lij},b_{li},c_l, $d \in L^\infty(\Omega)$, $a_{lij} \ge \alpha_{lij} > 0$,$b_{li} \ge \beta_{lij} > 0$,$c_l \ge 0, d \ge 0$.

3.1 The WF of the Proposed System

The WF of ((1)-(4)) is obtained by integrating both sides of ((1)-(2)) after multiplying them by the test function $v_i \in V = H_0^1(\Omega), (i = 1,2)$, resp., and then applying the generalized Green's theorem for the terms which have the derivative of 2nd order with employing ((3)-(4)), once get

$$\int_\Omega \sum_{i,j=1}^2 a_{1ij} \frac{\partial y_1}{\partial x_i}\frac{\partial v_1}{\partial x_j} dx + \int_\Omega \sum_{i=1}^2 b_{1i}(x)\frac{\partial y_1}{\partial x_i} v_1 dx + \int_\Omega c_1(x) y_1 v_1 dx - \int_\Omega d(x) y_2 v_1 dx = \int_\Omega f_1 v_1 dx, \forall v_1 \in H_0^1(\Omega), \tag{5}$$

and

$$\int_\Omega \sum_{i,j=1}^2 a_{2ij} \frac{\partial y_2}{\partial x_i}\frac{\partial v_2}{\partial x_j} dx + \int_\Omega \sum_{i=1}^2 b_{2i}(x)\frac{\partial y_2}{\partial x_i} v_2 dx + \int_\Omega c_2(x) y_2 v_2 dx + \int_\Omega d(x) y_1 v_2 dx = \int_\Omega f_2 v_2 dx, \quad \forall v_2 \in H_0^1(\Omega). \tag{6}$$

Equalities (5) and (6) can be also expressed as

$$a_1(y_1, v_1) + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d y_2, v_1)_\Omega = (f_1, v_1)_\Omega, \tag{7}$$

$$a_2(y_2, v_2) + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d y_1, v_2)_\Omega = (f_2, v_2)_\Omega. \tag{8}$$

By collecting (7) and (8)

$$a_1(y_1, v_1) + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d y_2, v_1)_\Omega + a_2(y_2, v_2) + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d y_1, v_2)_\Omega = (f_1, v_1)_\Omega + (f_2, v_2)_\Omega . \tag{9}$$

Or

$$a(\vec{y}, \vec{v}) = F(\vec{v}), \tag{10}$$

where

$$a(\vec{y}, \vec{v}) = a_1(y_1, v_1) + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d y_2, v_1)_\Omega + a_2(y_2, v_2) + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d y_1, v_2)_\Omega, \tag{10a}$$

$$F(\vec{v}) = (f_1, v_1)_\Omega + (f_2, v_2)_\Omega \tag{10b}$$

with

$$a_l(y_l, v_l) = \sum_{i,j=1}^2 a_{lij}(x) \frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}, \; b_l(y_l, v_l) = \sum_{i=1}^2 b_{li}(x)\frac{\partial y_l}{\partial x_i} v_l, \; l = 1,2 . \tag{10c}$$

3.2 Solvability of the CEPDEs

The following theorem deals with the solvability of WF (10).

Theorem (3.2.1): The WF (10) of the CEPDEs has a unique couple solution(COS) $\vec{y} = (y_1, y_2) \in \left(H_0^1(\Omega)\right)^2$.

Proof: The proof will be upon using the Theorem 2.2 (LMTH), and it starts by checking the coercivity of the $a(\vec{y}, \vec{v})$ in the LHS of (10), as

$$a(\vec{y}, \vec{y}) = a_1(y_1, y_1) + b_1(y_1, y_1) + (c_1 y_1, y_1)_\Omega - (d y_2, y_1)_\Omega + a_2(y_2, y_2) + b_2(y_2, y_2) + (c_2 y_2, y_2)_\Omega + (d y_1, y_2)_\Omega ,$$

$$a(\vec{y}, \vec{y}) = \int_\Omega \sum_{i,j=1}^2 a_{1ij}(x) \frac{\partial y_1}{\partial x_i}\frac{\partial y_1}{\partial x_j} dx + \int_\Omega \sum_{i=1}^2 b_{1i}(x)\frac{\partial y_1}{\partial x_i} y_1 dx + \int_\Omega c_1(x) y_1 y_1 dx +$$

$$\int_\Omega \sum_{i,j=1}^2 a_{2ij}(x) \frac{\partial y_2}{\partial x_i}\frac{\partial y_2}{\partial x_j} dx + \int_\Omega \sum_{i=1}^2 b_{2i}(x)\frac{\partial y_2}{\partial x_i} y_2 dx + \int_\Omega c_2(x) y_2 y_2 dx,$$

Since $a_{lij} \geq \alpha_{lij} > 0$, $(\forall\, l, \mathrm{i}, \mathrm{j} = 1,2)$, with setting $\frac{\partial y_l}{\partial x_i} y_l = \frac{1}{2}\frac{\partial}{\partial x_i}({y_l}^2)$, in the second and fifth terms, i.e.

$$a(\vec{y}, \vec{y}) \geq \alpha_{1ij} \sum_{i=1}^2 \int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \sum_{i=1}^2 \int_\Omega b_{1i}(x) \frac{1}{2}\frac{\partial}{\partial x_i}({y_1}^2) dx + \int_\Omega c_1(x)|y_1|^2 dx + \alpha_{2ij} \sum_{i=1}^2 \int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx + \sum_{i=1}^2 \int_\Omega b_{2i}(x) \frac{1}{2}\frac{\partial}{\partial x_i}({y_2}^2) dx + \int_\Omega c_2(x)|y_2|^2 dx \; . \tag{11}$$

Letting $\alpha = \min(\alpha_{lij})$, $\forall\, l, \mathrm{i}, \mathrm{j} = 1,2$, then using integrating by parts in the second and fifth terms on the RHS of (11), to get

$$a(\vec{y}, \vec{y}) \geq \alpha \left(\sum_{i=1}^2 \int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \sum_{i=1}^2 \int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx\right) + \int_\Omega \left(c_1(x) - \frac{1}{2}\sum_{i=1}^2 \frac{\partial b_{1i}}{\partial x_i}\right)|y_1|^2 dx + \int_\Omega \left(c_2(x) - \frac{1}{2}\sum_{i=1}^2 \frac{\partial b_{2i}}{\partial x_i}\right)|y_2|^2 dx.$$

Choose b_{li} and c_l for $l, i = 1,2$ such that the following is held

$$c_l(x) - \frac{1}{2}\sum_{i=1}^2 \frac{\partial b_{li}}{\partial x_i} \geq 0, \; x \in \bar{\Omega}, \tag{12}$$

which gives

$$a(\vec{y}, \vec{y}) \geq \alpha \sum_{i=1}^2 \|\Delta y_i\|^2_{\left(L^2(\Omega)\right)^2}. \tag{13}$$

Then by virtue of the Lemma2.3 (PFI), to get

$$a(\vec{y}, \vec{y}) \geq \frac{\alpha}{C_*} \sum_{i=1}^2 \|y_i\|^2_{\left(L^2(\Omega)\right)^2}. \tag{14}$$

Summing (13) and (14), letting $C_0 = \min(\alpha, \frac{\alpha}{C_*})$, one has

$a(\vec{y},\vec{y}) \geq C_0 \sum_{i=1}^{2}(\|y_i\|^2_{\left(L^2(\Omega)\right)^2} + \|\Delta y_i\|^2_{\left(L^2(\Omega)\right)^2}) = C_0 \sum_{i=1}^{2} \|y_i\|^2_{H_0^1}$.

Hence

$$a(\vec{y},\vec{y}) \geq C_0\|\vec{y}\|^2_{\left(H_0^1(\Omega)\right)^2} \tag{15}$$

Also, since $a_{lij}(x), b_{li}(x), c_l(x), d(x) \in L^\infty(\Omega), \forall l, i = 1,2$, so it flows

$$\left|\int_\Omega \sum_{i,j=1}^{2} a_{lij}(x)\ \frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}dx\right| \leq \sum_{i,j=1}^{2} \mu_l \left|\int_\Omega \frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}dx\right| \quad , \quad \text{with } \mu_l = \max_{x\in\bar{\Omega}}\left|a_{lij}(x)\right| , \tag{16}$$

$$\left|\int_\Omega \sum_{i=1}^{2} b_{li}(x)\ \frac{\partial y_l}{\partial x_i}v_l dx\right| \leq \sum_{i=1}^{2} \bar{\mu}_l \left|\int_\Omega \frac{\partial y_l}{\partial x_i}v_l dx\right| \quad , \text{ with } \quad \bar{\mu}_l = \max_{x\in\bar{\Omega}}\left|b_{li}(x)\right| \ , \tag{17}$$

$$\left|\int_\Omega c_l(x)y_l v_l\right| \leq \bar{\bar{\mu}}_l\left|\int_\Omega y_l v_l\, dx\right|, \text{ with } \bar{\bar{\mu}}_l = \max_{x\in\bar{\Omega}}|c_l(x)|, \tag{18}$$

$$\left|\int_\Omega d(x)y_{3-l} v_l\right| \leq \bar{\bar{\bar{\mu}}}_l\left|\int_\Omega y_{3-l} v_l\, dx\right|, \text{ with } \bar{\bar{\bar{\mu}}}_l = \max_{x\in\bar{\Omega}}|d(x)|, \tag{19}$$

Then, by inequalities ((16)-(19))

$$|a(\vec{y},\vec{v})| \leq \sum_{l=1}^{2}\sum_{i,j=1}^{2} \mu_l \left|\int_\Omega \frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}\ dx\right| + \sum_{l=1}^{2}\sum_{i,j=1}^{2} \bar{\mu}_l \left|\int_\Omega \frac{\partial y_l}{\partial x_i} v_l\ dx\right| + \sum_{l=1}^{2} \bar{\bar{\mu}}_l\left|\int_\Omega y_l v_l\, dx\right| + \sum_{l=1}^{2} \bar{\bar{\bar{\mu}}}_l\left|\int_\Omega y_{3-l} v_l\, dx\right| .$$

By the Cauchy –Schwarz (CS) inequality [21] , one gets

$$|a(\vec{y},\vec{v})| \leq \mu\left\{\sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega \left|\frac{\partial v_1}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega \left|\frac{\partial v_2}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \sum_{i=1}^{2}\left(\int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}}\right\}, \mu = \max\{\mu_l, \bar{\mu}_l, \bar{\bar{\mu}}_l, \bar{\bar{\bar{\mu}}}_l\}, \forall l, i = 1,2,$$

$$|a(\vec{y},\vec{v})| \leq \mu\left(\left\{\left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}} + \sum_{j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_1}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}}\right\} \times \left\{\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial v_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\right\} + \left\{\left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_2}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}}\right\} \times \left\{\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial v_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}}\right\} + \left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}}\right). \tag{20}$$

The RHS of (20) can be rewritten as

$$|a(\vec{y},\vec{v})| \leq 4\mu\left\{\int_\Omega |y_1|^2 dx + \sum_{j=1}^{2}\int_\Omega \left|\frac{\partial y_1}{\partial x_j}\right|^2 dx\right\}^{\frac{1}{2}} \times \left\{\int_\Omega |v_1|^2 dx + \sum_{i=1}^{2}\int_\Omega \left|\frac{\partial v_1}{\partial x_i}\right|^2 dx\right\}^{\frac{1}{2}} + 4\mu\left\{\int_\Omega |y_2|^2\, dx + \sum_{j=1}^{2}\int_\Omega \left|\frac{\partial y_2}{\partial x_j}\right|^2 dx\right\}^{\frac{1}{2}} \times \left\{\int_\Omega |v_2|^2\, dx + \sum_{i=1}^{2}\int_\Omega \left|\frac{\partial v_2}{\partial x_i}\right|^2 dx\right\}^{\frac{1}{2}} + \mu\left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \mu\left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} \leq 4\mu\left[\|y_1\|_{H_0^1(\Omega)}\|v_1\|_{H_0^1(\Omega)} + \|y_2\|_{H_0^1(\Omega)}\|v_2\|_{H_0^1(\Omega)}\right] + \mu\|y_2\|_{H_0^1(\Omega)}\|v_1\|_{H_0^1(\Omega)} + \mu\|y_1\|_{H_0^1(\Omega)}\|v_2\|_{H_0^1(\Omega)} \leq \rho\|\vec{y}\|_{\left(H_0^1(\Omega)\right)^2}\|\vec{v}\|_{\left(H_0^1(\Omega)\right)^2}$$

So, by letting $\rho = 10\mu$, one concludes that

$$a(\vec{y},\vec{v}) \leq \rho\ \|\vec{y}\|_{\left(H_0^1(\Omega)\right)^2}\|\vec{v}\|_{\left(H_0^1(\Omega)\right)^2}. \tag{21}$$

It is clear that $F(\vec{v})$ in the RHS of (10) is a linear functional on $\left(H_0^1(\Omega)\right)^2$. It remains to show that it is bounded, so by the CS inequality

$$\begin{aligned}|F(\vec{v})| &= \sum_{l=1}^{2}[|(f_l, v_l)_\Omega|] \\ &\leq \sum_{l=1}^{2}[\|f_l(x)\|_{L^2(\Omega)}\|v_l\|_{L^2(\Omega)}] \\ &\leq \sum_{l=1}^{2}[\gamma_l\|v_l\|_{L^2(\Omega)}], \quad \|f_l(x)\|_{L^2(\Omega)} \leq \gamma_l, \gamma = \max(\gamma_l), \forall l = 1,2 \\ &\leq \gamma\|\vec{v}\,\|_{\left(L^2(\Omega)\right)^2} \leq \gamma\|\vec{v}\,\|_{\left(H_0^1(\Omega)\right)^2},\end{aligned} \tag{22}$$

Then, by the Theorem2.2, the WF (10) has unique COS $\vec{y} = (y_1, y_2) \in \left(H_0^1(\Omega)\right)^2$.

Corollary (3.2.1)

The WF of the CEPDEs ((1) -(2)), with the following nonhomogeneous Dirichlet BCs(NHDBCs)

$$y_1 = g_1 \quad on \ \ \Gamma, \tag{23}$$

$y_2 = g_2 \quad on \quad \Gamma,$ (24)

Has a unique COS $\vec{y} = (y_1, y_2) \in \left(H^1(\Omega)\right)^2$, where $g_1(x), g_2(x) \in L^2(\Gamma), \vec{g} = (g_1, g_2)$.

Proof:

Consider, $\vec{w} = (w_1, w_2),\ \vec{w} \in \left(H^2(\Omega)\right)^2$, s.t $\vec{w} = \vec{g}$ on Γ.

Let $\vec{z} = (z_1, z_2) \in \left(H_0^2(\Omega)\right)^2$, and $\vec{y} = \vec{w} + \vec{z}$, then problem ((1),(2)) with the NHDBCs (23),(24) are equivalent to:

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{1ij}(x)\frac{\partial z_1}{\partial x_j}\right) + \sum_{i=1}^{2} b_{1i}(x)\frac{\partial z_1}{\partial x_i} + c_1(x)z_1 - d(x)z_2 = h_1(x), \quad (25)$$

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{2ij}(x)\frac{\partial z_2}{\partial x_j}\right) + \sum_{i=1}^{2} b_{2i}(x)\frac{\partial z_2}{\partial x_i} + c_2(x)z_2 + d(x)z_1 = h_2(x), \quad (26)$$

$z_1 = 0 \quad on \quad \Gamma,$ (27)

$z_2 = 0 \quad on \quad \Gamma,$ (28)

Where $h_l(x) = f_l(x) + \sum_{ij=1}^{2} \frac{\partial}{\partial x_i}\left(a_{lij}(x)\frac{\partial w_l}{\partial x_j}\right) - \sum_{i=1}^{2} b_{li}(x)\frac{\partial w_l}{\partial x_i} - c_l(x)w_l + (-1)^l d(x)w_{3-l}$, $\forall l = 1,2$.

The WF which results from ((25)-(28)) is similar to the WF (10) with two different, the first one of them is that z_l (for $l = 1,2$) is instead of y_l in the LHS, and the second is that the RHS will be in the following form ($\forall v_l \in H_0^1(\Omega), l = 1,2$),

$$(h_1, v_1)_\Omega = (f_1, v_1)_\Omega + a_1(w_1, v_1) - b_1(w_1, v_1) - (c_1 w_1, v_1)_\Omega + (dw_2, v_1)_\Omega, \quad (29)$$
$$(h_2, v_2)_\Omega = (f_2, v_2)_\Omega + a_2(w_2, v_2) - b_2(w_2, v_2) - (c_2 w_2, v_2)_\Omega - (dw_1, v_2)_\Omega, \quad (30)$$

Collecting (29) and (30), gives

$$H(\vec{v}) = (h_1, v_1)_\Omega + (h_2, v_2)_\Omega, \quad (31)$$

i.e

$$a(\vec{y}, \vec{v}) = H(\vec{v}), \quad (32)$$

In this case, to apply Theorem2.2, it is enough to show that the RHS of WF (32) is continuous. To do this, the absolute value for both sides of (31) is taken, to get

$|H(\vec{v})| \le \sum_{i=1}^{2}[\,|(f_l, v_l)_\Omega| + |a_l(w_l, v_l)| + |b_l(w_l, v_l)| + |(c_l w_l, v_l)_\Omega| + |(dw_{3-l}, v_l)_\Omega|] = |F(\vec{v})| + |a(\vec{y}, \vec{v})|$, (by using (10a) and (10b)).

Then, by applying the results in (21) and (22) with $\vec{w}$ instead of $\vec{y}$ to get

$$|H(\vec{v})| \le \gamma \|\vec{v}\|^2_{\left(H_0^1(\Omega)\right)^2} + \rho\, \|\vec{w}\|_{\left(H_0^1(\Omega)\right)^2} \|\vec{v}\|_{\left(H_0^1(\Omega)\right)^2}.$$

Since $\|\vec{w}\|_{\left(H_0^1(\Omega)\right)^2} \le \lambda$, then the above inequality gives

$$|H(\vec{v})| \le \bar{\gamma}\|\vec{v}\|_{\left(H_0^1(\Omega)\right)^2}, \text{ where } \bar{\gamma} = \gamma + \rho\lambda. \quad (33)$$

By Theorem(2.2), the WF(32) has a unique COS $\vec{z} = (z_1, z_2) \in \left(H_0^1(\Omega)\right)^2$, thus there is a unique COS $\vec{y} = (y_1, y_2) \in \left(H^1(\Omega)\right)^2$.

4 Solvability of the CEPDEs with the NBCs

In this section the solvability of the CEPDEs with Neumann BCs (NBCs) is studied.

Theorem (4.1)

The proposed CEPDEs ((1)-(2)), with the following nonhomogenous NBCs (NHNBCs)

$$\frac{\partial y_1}{\partial n_1} = \sum_{i,j=1}^{2} a_{1ij}\frac{\partial y_1}{\partial x_i}\ cos(n_1, x_j) = g_1 \text{ on } \Gamma, \quad (34)$$

$$\frac{\partial y_2}{\partial n_2} = \sum_{i,j=1}^{2} a_{2ij}\frac{\partial y_2}{\partial x_i}\ cos(n_2, x_j) = g_2 \text{ on } \Gamma, \quad (35)$$

Has a unique COS $\vec{y} = (y_1, y_2) \in \left(H^1(\Omega)\right)^2$ with $(g_1, g_2) \in (L^2(\Gamma))^2$ and n_ℓ (for $\ell = 1,2$) is an outer normal vector on Γ and (n_ℓ, x_j) is the angle between n_ℓ and the $x_j - axis$.

Proof:

The WF, which results from ((1)-(2)) with the NHNBCs (33)-(34)) is similar to the WF (10) with one difference that the RHS, in this case becomes

$$F(\vec{v}) = \sum_{l=1}^{2}[\int_\Omega f_l(x)v_l dx + \int_\Gamma g_l v_l d\gamma], \forall v_l \in H^1(\Omega). \quad (36)$$

Hence, to apply the LMTH, and since the $a(\vec{y}, \vec{v})$ in the LHS of(36) is the same as in the LHS of (10), and the coercivity and the boundedness are proved there, it remains to prove the RHS of (36) is bounded, so by CS inequality and the trace "operator(TRO)(Theorem2.5),"

$|F(\vec{v})| \le \sum_{i=1}^{2}[|(f_i, v_i)_\Omega| + |(g_i, v_i)_\Gamma|] \le \sum_{i=1}^{2}[\|f_i(x)\|_{L^2(\Omega)}\|v_i\|_{L^2(\Omega)} + \|g_i(x)\|_{L^2(\Gamma)}\|v_i\|_{L^2(\Gamma)}] \le \sum_{i=1}^{2}[\mu_i\|v_i\|_{H^1(\Omega)} + \bar{\mu}_i\bar{\bar{\mu}}_i\|v_i\|_{H^1(\Omega)}]$, where $\|f_l(x)\|_{L^2(\Omega)} \le \mu_l$, $\|g_l(x)\|_{L^2(\Gamma)} \le \bar{\mu}_l \quad \le \check{C}\, \|\vec{v}\|_{\left(H^1(\Omega)\right)^2}$, where $\check{C} = 2max(\mu_l, \bar{\mu}_l\bar{\bar{\mu}}_l)$. (37)

The proof is complete.

Remark 4.1: If the NBCs ((34)-(35)) are homogeneous, then the RHS in (37) will be similar to (22) and is proved by a similar way which was used in the proof of Theorem 3.2.1.

5 Solvability of the CEPDEs with the RBCs

In this section, the solvability of the CEPDEs with the RBCs is studied.

Theorem (5.1)

The proposed system in ((1)-(2)), with nonhomogenous Robin BCs (NHRBCs)

$$\alpha_1 y_1 + \frac{\partial y_1}{\partial n_1} = \alpha_1 y_1 + \sum_{i,j=1}^{2} a_{1ij}\frac{\partial y_1}{\partial x_i}\, cos(n_1, x_j) = g_1 \quad on \quad \Gamma, \tag{38}$$

$$\alpha_2 y_2 + \frac{\partial y_2}{\partial n_2} = \alpha_2 y_2 + \sum_{i,j=1}^{2} a_{2ij}\frac{\partial y_2}{\partial x_i}\, cos(n_2, x_j) = g_2 \quad on \quad \Gamma, \tag{39}$$

Has unique COS $\vec{y} = (y_1, y_2) \in (H^1(\Omega))^2$, with $g_1, g_2 \in L^2(\Gamma)$ and $\alpha_1, \alpha_2 > 0$, and n_ℓ, (for $\ell = 1,2$) is as defined in Theorem 4.1.

Proof:

The WF of the CEPDES ((1)-(2)) with the NHRBCs is obtained by multiplying both sides of ((1)-(2)) and ((38)-(39)) by the test function $v_i \in H^1(\Omega), (i = 1,2)$, resp., then applying the generalized Green's theorem for the terms which have the 2nd order derivatives, once get:

$$\int_\Omega \sum_{i,j=1}^{2} a_{1ij}\frac{\partial y_1}{\partial x_i}\frac{\partial v_1}{\partial x_j}dx + \int_\Gamma \alpha_1 y_1 v_1 d\gamma + \int_\Omega \sum_{i=1}^{2} b_{1i}(x)\frac{\partial y_1}{\partial x_i}v_1 dx + \int_\Omega c_1(x) y_1 v_1 dx - \int_\Omega d(x) y_2 v_1 dx$$

$$= \int_\Omega f_1 v_1 dx + \int_\Gamma g_1 v_1 d\gamma, \ \forall v_1 \in H^1(\Omega). \tag{40}$$

$$\int_\Omega \sum_{i,j=1}^{2} a_{2ij}\frac{\partial y_2}{\partial x_i}\frac{\partial v_2}{\partial x_j}dx + \int_\Gamma \alpha_2 y_2 v_2 d\gamma + \int_\Omega \sum_{i=1}^{2} b_{2i}(x)\frac{\partial y_2}{\partial x_i}v_2 dx + \int_\Omega c_2(x) y_2 v_2 dx + \int_\Omega d(x) y_1 v_2 dx = \int_\Omega f_2 v_2 dx + \int_\Gamma g_2 v_2 d\gamma, \forall v_2 \in H^1(\Omega). \tag{41}$$

Equations (40) and (41) can be expressed also as

$$a_1(y_1, v_1) + (\alpha_1 y_1, v_1)_\Gamma + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d y_2, v_1)_\Omega = (f_1, v_1)_\Omega + (g_1, v_1)_\Gamma, \tag{42}$$

and

$$a_2(y_2, v_2) + (\alpha_2 y_2, v_2)_\Gamma + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d y_1, v_2)_\Omega = (f_2, v_2)_\Omega + (g_2, v_2)_\Gamma, \tag{43}$$

Adding (42) and (43), to obtain

$$a_1(y_1, v_1) + (\alpha_1 y_1, v_1)_\Gamma + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d y_2, v_1)_\Omega + a_2(y_2, v_2) + (\alpha_2 y_2, v_2)_\Gamma + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d y_1, v_2)_\Omega = (f_1, v_1)_\Omega + (g_1, v_1)_\Gamma + (f_2, v_2)_\Omega + (g_2, v_2)_\Gamma . \tag{44}$$

Or

$$a(\vec{y}, \vec{v}) = F(\vec{v}), \tag{45}$$

where $a(\vec{y}, \vec{v}) = a_1(y_1, v_1) + (\alpha_1 y_1, v_1)_\Gamma + b_1(y_1, v_1) + (c_1 y_1, v_1)_\Omega - (d y_2, v_1)_\Omega + a_2(y_2, v_2) + (\alpha_2 y_2, v_2)_\Gamma + b_2(y_2, v_2) + (c_2 y_2, v_2)_\Omega + (d y_1, v_2)_\Omega$,

with $a_l(y_l, v_l), b_l(y_l, v_l)$ are as defined in (10-c), $(\alpha_l y_l, v_l)_\Gamma = \int_\Gamma \alpha_l y_l v_l d\gamma$, $l = 1,2$, and $F(\vec{v})$ is as defined in (36).

Now, the proof of the coercivity and boundedness of $a(\vec{y}, \vec{v})$ in LHS of (45), as follows

$$a(\vec{y}, \vec{y}) = \int_\Omega \sum_{i,j=1}^{2} a_{1ij}\frac{\partial y_1}{\partial x_i}\frac{\partial y_1}{\partial x_j}dx + \int_\Gamma \alpha_1 y_1 y_1 d\gamma + \int_\Omega \sum_{i=1}^{2} b_{1i}(x)\frac{\partial y_1}{\partial x_i}y_1 dx + \int_\Omega c_1(x) y_1 y_1 dx - \int_\Omega d(x) y_2 y_1 dx + \int_\Omega \sum_{i,j=1}^{2} a_{2ij}\frac{\partial y_2}{\partial x_i}\frac{\partial y_2}{\partial x_j}dx + \int_\Gamma \alpha_2 y_2 y_2 d\gamma + \int_\Omega \sum_{i=1}^{2} b_{2i}(x)\frac{\partial y_2}{\partial x_i}y_2 dx + \int_\Omega c_2(x) y_2 y_2 dx + \int_\Omega d(x) y_1 y_2 dx .$$

Setting $\frac{\partial y_l}{\partial x_i} y_l = \frac{1}{2}\frac{\partial}{\partial x_i}(y_l{}^2), l = 1,2,$ for the third and eighth terms, then it becomes

$$a(\vec{y}, \vec{y}) \geq \hat{c}_1 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \hat{c}_2 \int_\Gamma |y_1|^2 d\gamma + \sum_{i=1}^{2} \int_\Omega b_{1i}(x)\frac{1}{2}\frac{\partial}{\partial x_i}(y_1{}^2)\, dx + \int_\Omega c_1(x)|y_1|^2 dx + \hat{c}_3 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx + \hat{c}_4 \int_\Gamma |y_2|^2 d\gamma + \sum_{i=1}^{2} \int_\Omega b_{2i}(x)\frac{1}{2}\frac{\partial}{\partial x_i}(y_2{}^2)\, dx + \int_\Omega c_2(x)|y_2|^2 dx. \tag{46}$$

By using integration by parts on the third and the seventh terms in the RHS of (46), we obtain

$$a(\vec{y}, \vec{y}) \geq \hat{c}_1 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \hat{c}_2 \int_\Gamma |y_1|^2 d\gamma + \sum_{i=1}^{2} \int_\Gamma \frac{b_{1i}(x)}{2} |y_1|^2 d\gamma + \int_\Omega \left(c_1(x) - \frac{1}{2}\sum_{i=1}^{2}\frac{\partial b_{1i}}{\partial x_i}\right)|y_1|^2 dx + \hat{c}_4 \int_\Gamma |y_2|^2 d\gamma + \hat{c}_3 \sum_{i=1}^{2} \int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx + \sum_{i=1}^{2} \int_\Gamma \frac{b_{2i}(x)}{2} |y_2|^2 d\gamma + \int_\Omega \left(c_2(x) - \frac{1}{2}\sum_{i=1}^{2}\frac{\partial b_{2i}}{\partial x_i}\right)|y_2|^2 dx. \tag{47}$$

Using (12) in (47) and $b_{li} \geq 0$ for $l, i = 1,2$, then (47) becomes

$$a(\vec{y},\vec{y}) \geq \hat{c}_1 \sum_{i=1}^{2}\int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx + \hat{c}_3 \sum_{i=1}^{2}\int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx + \hat{c}_2 \int_\Gamma |y_1|^2\, d\gamma + \hat{c}_4 \int_\Gamma |y_2|^2\, d\gamma ,$$
$$= C_0\left(\sum_{i=1}^{2}\int_\Omega |\nabla y_i|^2\, dx + \sum_{i=1}^{2}\int_\Gamma |y_i|^2\, d\gamma\right), \quad \tau = \min(\hat{c}_2,\hat{c}_4), \quad C_0 = \min(\hat{c}_1,\hat{c}_3,\tau). \tag{48}$$

Now, from the assumption on Ω, let $\Gamma_1 \subset \Gamma$ be a measureable set with $|\Gamma_1| > 0$ then by Theorem2.4, there exist constants $\sigma_1(\Gamma_1), \sigma_2(\Gamma_1) > 0$ which are independent of $\vec{y} \in \left(H^1(\Omega)\right)^2$ s.t.

$$a(\vec{y},\vec{y}) \geq C_0\left(\sum_{i=1}^{2}\int_\Omega |\nabla y_i|^2\, dx + \sum_{i=1}^{2}\int_\Gamma |y_i|^2\, d\gamma\right)$$

$$= \sigma_3\left(\sum_{i=1}^{2}\int_\Omega |\nabla y_i|^2\, dx + \sum_{i=1}^{2}\int_{\Gamma_1} |y_i|^2\, d\gamma\right) \geq \sigma_3 \|\vec{y}\|^2_{\left(H^1(\Omega)\right)^2}, \quad \sigma_3 = C_0 \min(\sigma_1,\sigma_2). \tag{49}$$

On the other hand, since $a_{lij}\alpha_l \in L^\infty(\Omega)$ for $(\forall l, i = 1,2)$ then

$$\left|\int_\Gamma \sum_{=1}^{2} \alpha_l y_l v_l d\gamma\right| \leq \sum_{l=1}^{2}\lambda_l \left|\int_\Gamma y_l v_l d\gamma\right|, \quad \lambda_l = \max_{l=1,2}|\alpha_l| \tag{50}$$

Using (50) beside the inequalities ((16) -(19)), to obtain

$$|a(\vec{y},\vec{v})| \leq \sum_{i,j=1}^{2}\mu_1 \left|\int_\Omega \frac{\partial y_1}{\partial x_i}\frac{\partial v_1}{\partial x_j}\, dx\right| + \sum_{l=1}^{2}\lambda_l \left|\int_\Gamma y_l v_l d\gamma\right| + \sum_{i,j=1}^{2}\mu_2 \left|\int_\Omega \frac{\partial y_2}{\partial x_i}\frac{\partial v_2}{\partial x_j}\, dx\right| + \sum_{i=1}^{2}\bar{\mu}_1 \left|\int_\Omega \frac{\partial y_1}{\partial x_i} v_1\, dx\right| + \sum_{i=1}^{2}\bar{\mu}_2 \left|\int_\Omega \frac{\partial y_2}{\partial x_i} v_2\, dx\right| + \sum_{l=1}^{2}\bar{\bar{\mu}}_l \left|\int_\Omega y_l v_l\, dx\right| + \sum_{l=1}^{2}\bar{\bar{\bar{\mu}}}_l \left|\int_\Omega y_{3-l} v_l\, dx\right| .$$

By the CS inequality for each term in the above inequality, to get that

$$|a(\vec{y},\vec{v})| \leq \mu \Bigg\{ \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\frac{\partial v_1}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Gamma |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Gamma |v_1|^2 dx\right)^{\frac{1}{2}} +$$
$$\sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\frac{\partial v_2}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Gamma |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Gamma |v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}}$$
$$+ \sum_{i=1}^{2}\left(\int_\Omega \left|\frac{\partial y_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \sum_{i=1}^{2}\left(\int_\Omega \left|\frac{\partial y_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} +$$
$$\left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}}\Bigg\}, \tag{51}$$

where $\mu = \max\{\mu_l, \lambda_l, \bar{\mu}_l, \bar{\bar{\mu}}_l, \bar{\bar{\bar{\mu}}}_l\}, \forall l, i = 1,2.$

Using the Theorem2.5 for the second and the fourth term in the above inequality and then rewriting the resulting inequality in another form, it becomes

$$|a(\vec{y},\vec{v})| \leq \mu \Bigg(\left\{ \left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}} + \sum_{j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_1}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} \right\} \times \left\{ \left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial v_1}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \right\} +$$
$$\left\{ \left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial y_2}{\partial x_j}\right|^2 dx\right)^{\frac{1}{2}} \right\} \times \left\{ \left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}} + \sum_{i,j=1}^{2}\left(\int_\Omega \left|\frac{\partial v_2}{\partial x_i}\right|^2 dx\right)^{\frac{1}{2}} \right\} + \left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} +$$
$$\left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}}\Bigg) +$$
$$\bar{c}_1\left(\int_\Omega |y_1|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_1|^2 dx\right)^{\frac{1}{2}} + \bar{c}_2\left(\int_\Omega |y_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_2|^2 dx\right)^{\frac{1}{2}}\Bigg), \tag{52}$$

$$|a(\vec{y},\vec{v})| \leq \mu_3\big[\|y_1\|_{H^1(\Omega)}\|v_1\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)} + \|y_2\|_{H^1(\Omega)}\|v_1\|_{H^1(\Omega)} + \|y_1\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)} + \bar{c}_1\|y_1\|_{H^1(\Omega)}\|v_1\|_{H^1(\Omega)} + \bar{c}_2\|y_2\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)}\big],$$
where $\mu_3 = \mu\max(1, \max_{1\leq l\leq 2}(1+\bar{c}_l))$, $\leq \rho\|\vec{y}\|_{\left(H^1(\Omega)\right)^2}\|\vec{v}\|_{\left(H^1(\Omega)\right)^2}$, $\rho = 6\mu_3$. (53)

In this case, it is also clear that $F(\vec{v})$ in the RHS of (45)is a linear functional on $\left(H^1(\Omega)\right)^2$. It remains to show that it is bounded. In this case since $F(\vec{v})$ is the same in the RHS of (36), and it was proved there is bounded, hence by the theorem2.2. The WF (45) has a unique solution $\vec{y} \in \left(H^1(\Omega)\right)^2$.

Corollary (5.1)

The CEPDEs ((1)-(2)), with the homogenous Robin BCs (HRBCs)

$$\alpha_1 y_1 + \frac{\partial y_1}{\partial n_1} = \alpha_1 y_1 + \sum_{i,j=1}^{2} a_{1ij}\frac{\partial y_1}{\partial x_i}\cos(n_1, x_j) = 0 \quad on \quad \Gamma, \tag{54}$$

$$\alpha_2 y_2 + \frac{\partial y_2}{\partial n_2} = \alpha_2 y_2 + \sum_{i,j=1}^{2} a_{2ij}\frac{\partial y_2}{\partial x_i}\cos(n_2, x_j) = 0 \quad on \quad \Gamma, \tag{55}$$

Has unique COS $\vec{y}=(y_1,y_2)\in\left(H_0^2(\Omega)\right)^2$, with $\alpha_1,\alpha_2$and n_ℓ, (for $\ell=1,2$) are as defined in Theorem(5.1)

Proof:

The WF of the CEPDES ((1)-(2))and ((54)-(55)) will be similar to the WF in Theorem (4.1) with one difference, which is the RHS of WF(10) will be missing the two terms $\int_\Gamma g_1 v_1 d\gamma$ and $\int_\Gamma g_2 v_2 d\gamma$ in this case. Hence the existence of a unique COS in this case will be obtained directly after applying Theorem4.1

6 Solvability of the CEPDEs with the MBCs

In this section, the solvability of the CEPDEs with the MBCs is studied

Theorem (6.1)

The CEPDEs ((1)-(2)) with the following Mixed BCs (MBCs),

$$y_l=g_{lD} \quad \text{on} \quad \Gamma_D\,,\ l=1,2 \tag{56}$$

$$\frac{\partial y_l}{\partial n_l}=g_{lN} \quad \text{on} \quad \Gamma_N\,,\ l=1,2\,, \tag{57}$$

Has a unique COS $\vec{y}=(y_1,y_2)\in\left(H^2(\Omega)\right)^2$, where the boundary Γ of Ω be split into two nonempty disjoint open sets Γ_D and Γ_N with $\vec{g}_l=(g_{lD},g_{lN})\in L^2(\Gamma_D)\times L^2(\Gamma_N)$, and n_l is a unite outer vector normal on the boundaryΓ_N, $\Gamma=\Gamma_D\cup\Gamma_N$, and $\Gamma_D\cap\Gamma_N=\emptyset$.

Proof:

To find the WF of the CEPDES ((1)-(2)) with the MBCs ((58)-(59)), at first by introducing a new extending function$\vec{w}\in\left(H^2(\Omega)\right)^2$, s.t. $\vec{w}=\vec{\hat{g}}_l=(\hat{g}_{1D},\hat{g}_{2D})\in L^2(\Gamma)\times L^2(\Gamma)$, then the BC(58) is reduced to a homogeneous type through setting $\vec{y}=\vec{z}+\vec{w}$. Now let $\vec{V}=\{\vec{z}\in H^1(\Omega),\vec{z}=0 \ on\ \Gamma_D\times\Gamma_D\}$, then ((1)-(2)) can be rewrite as,

$$A_1z_1+B_1z_1+c_1(x)z_1-d(x)z_2=f_1(x)-A_1w_1-B_1w_1-c_1(x)w_1+d(x)w_2 \tag{58}$$

$$A_2z_2+B_2z_2+c_2(x)z_2+d(x)z_1=f_2(x)-A_2w_2-B_2w_2-c_2(x)w_2-d(x)w_1 \tag{59}$$

$$z_l=0 \quad \text{on} \quad \Gamma_D\,,\ l=1,2\,, \tag{60}$$

$$\frac{\partial(z_l+w_l)}{\partial n_l}=g_{lN} \quad \text{on } \Gamma_N\,,\ l=1,2 \tag{61}$$

where $A_l(.)=-\sum_{i,j=1}^2\frac{\partial}{\partial x_i}\left(a_{lij}(x)\frac{\partial(.)}{\partial x_j}\right)$, $B_l(.)=\sum_{i=1}^2 b_{li}(x)\frac{\partial(.)}{\partial x_i}$, $l=1,2.$, $x=(x_1,x_2)\in\Omega$, $f_l(x)\in L^2(\Omega)$, and a_{lij},b_{li},c_l, $d\in L^\infty(\Omega)$, $a_{lij}\geq\alpha_{lij}>0$,$b_{li}\geq\beta_{lij}>0$,$c_l\geq0$, $d\geq0$.

Integrating both sides of ((58)-(59)) and ((60)-(61)) after multiplying of them by the test function, $v_i\in H^1(\Omega),(i=1,2)$, resp., to obtain

$$-\int_\Omega\sum_{i,j=1}^2\frac{\partial}{\partial x_i}\left(a_{1ij}(x)\frac{\partial z_1}{\partial x_j}\right)v_1\,dx+\int_\Omega\sum_{i=1}^2 b_{1i}(x)\frac{\partial z_1}{\partial x_i}v_1\,dx+\int_\Omega c_1(x)z_1v_1\,dx-\int_\Omega d(x)z_2v_1\,dx=\int_\Omega f_1(x)v_1dx+\int_\Omega\sum_{i,j=1}^2\frac{\partial}{\partial x_i}\left(a_{1ij}(x)\frac{\partial w_1}{\partial x_j}\right)v_1\,dx-\int_\Omega\sum_{i=1}^2 b_{1i}(x)\frac{\partial w_1}{\partial x_i}v_1\,dx-\int_\Omega c_1(x)w_1v_1\,dx+\int_\Omega d(x)w_2v_1\,dx, \tag{62}$$

$$-\int_\Omega\sum_{i,j=1}^2\frac{\partial}{\partial x_i}\left(a_{2ij}(x)\frac{\partial z_2}{\partial x_j}\right)v_2\,dx+\int_\Omega\sum_{i=1}^2 b_{2i}(x)\frac{\partial z_2}{\partial x_i}v_2\,dx+\int_\Omega c_2(x)z_2v_2\,dx+\int_\Omega d(x)z_1v_2\,dx=\int_\Omega f_2(x)v_2dx+\int_\Omega\sum_{i,j=1}^2\frac{\partial}{\partial x_i}\left(a_{2ij}(x)\frac{\partial w_2}{\partial x_j}\right)v_2\,dx-\int_\Omega\sum_{i=1}^2 b_{2i}(x)\frac{\partial w_2}{\partial x_i}v_2\,dx-\int_\Omega c_2(x)w_2v_2\,dx-\int_\Omega d(x)w_1v_2\,dx, \tag{63}$$

Applying the generalized Green's theorem for the terms which have the 2nd order derivatives, using the resulting BCs, once get:

$$\int_\Omega\sum_{i,j=1}^2 a_{1ij}(x)\frac{\partial z_1}{\partial x_j}\frac{\partial v_1}{\partial x_j}dx\,dx+\int_\Omega\sum_{i=1}^2 b_{1i}(x)\frac{\partial z_1}{\partial x_i}v_1\,dx+\int_\Omega c_1(x)z_1v_1\,dx-\int_\Omega d(x)z_2v_1\,dx=\int_\Omega f_1(x)v_1\,dx-\int_\Omega\sum_{i,j=1}^2 a_{1ij}(x)\frac{\partial w_1}{\partial x_j}\frac{\partial v_1}{\partial x_j}-\int_\Omega\sum_{i=1}^2 b_{1i}(x)\frac{\partial w_1}{\partial x_i}v_1\,dx-\int_\Omega c_1(x)w_1v_1\,dx+\int_\Omega d(x)w_2v_1\,dx+\int_{\Gamma_N}\sum_{i,j=1}^2 a_{1ij}(x)\frac{\partial(z_1+w_1)}{\partial x_j}v_1dx\,, \tag{64}$$

$$\int_\Omega\sum_{i,j=1}^2 a_{2ij}(x)\frac{\partial z_2}{\partial x_j}\frac{\partial v_2}{\partial x_j}dx\,dx+\int_\Omega\sum_{i=1}^2 b_{2i}(x)\frac{\partial z_2}{\partial x_i}v_2\,dx+\int_\Omega c_2(x)z_2v_2\,dx+\int_\Omega d(x)z_1v_2\,dx=\int_\Omega f_2(x)v_2\,dx-\int_\Omega\sum_{i,j=1}^2 a_{2ij}(x)\frac{\partial w_2}{\partial x_j}\frac{\partial v_2}{\partial x_j}-\int_\Omega\sum_{i=1}^2 b_{2i}(x)\frac{\partial w_2}{\partial x_i}v_2\,dx-\int_\Omega c_2(x)w_2v_2\,dx-\int_\Omega d(x)w_1v_2\,dx+\int_{\Gamma_N}\sum_{i,j=1}^2 a_{2ij}(x)\frac{\partial(z_2+w_2)}{\partial x_j}v_2dx\,. \tag{65}$$

Equations (64) and (65) can be expressed also as

$$a_1(z_1,v_1)+b_1(z_1,v_1)+(c_1z_1,v_1)_\Omega-(dz_2,v_1)_\Omega=(f_1,v_1)_\Omega-a_1(w_1,v_1)-b_1(w_1,v_1)-(c_1w_1,v_1)_\Omega+(dw_2,v_1)_\Omega+(g_{1N},v_1)_{\Gamma_N}. \tag{66}$$

And

$$a_2(z_2,v_2)+b_2(z_2,v_2)+(c_2z_2,v_2)_\Omega+(dz_1,v_2)_\Omega=(f_2,v_2)_\Omega-a_2(w_2,v_2)-b_2(w_2,v_2)-(c_2w_2,v_2)_\Omega-(dw_1,v_2)_\Omega+(g_{2N},v_2)_{\Gamma_N}. \tag{67}$$

Adding (66) and (67), to get

$$a_1(z_1,v_1)+b_1(z_1,v_1)+(c_1z_1,v_1)_\Omega-(dz_2,v_1)_\Omega+a_2(z_2,v_2)+b_2(z_2,v_2)+(c_2z_2,v_2)_\Omega+(dz_1,v_2)_\Omega=(f_1,v_1)_\Omega-a_1(w_1,v_1)-b_1(w_1,v_1)-(c_1w_1,v_1)_\Omega+$$

$$(dw_2, v_1)_\Omega + (f_2, v_2)_\Omega - a_2(w_2, v_2) - b_2(w_2, v_2) - (c_2 w_2, v_2)_\Omega - (dw_1, v_2)_\Omega + (g_{1N}, v_1)_{\Gamma_N} + (g_{2N}, v_2)_{\Gamma_N}. \quad (68)$$

Or

$$a(\vec{y}, \vec{v}) = \bar{F}(\vec{v}), \quad (69)$$

where $a(\vec{z}, \vec{v}) = a_1(z_1, v_1) + b_1(z_1, v_1) + (c_1 z_1, v_1)_\Omega - (dz_2, v_1)_\Omega + a_2(z_2, v_2) + b_2(z_2, v_2) + +(dz_1, v_2)_\Omega$,

$$\bar{F}(\vec{v}) = (f_1, v_1)_\Omega - a_1(w_1, v_1) - b_1(w_1, v_1) - (c_1 w_1, v_1)_\Omega + (dw_2, v_1)_\Omega + (f_2, v_2)_\Omega - a_2(w_2, v_2) - b_2(w_2, v_2) - (c_2 w_2, v_2)_\Omega - (dw_1, v_2)_\Omega + (g_{1N}, v_1)_{\Gamma_N} + (g_{2N}, v_2)_{\Gamma_N}.$$

Or $\bar{F}(\vec{v}) = F(\vec{v}) - a(\vec{w}, \vec{v}) + (g_{1N}, v_1)_{\Gamma_N} + (g_{2N}, v_2)_{\Gamma_N}$.

The coercivity and boundedness proof of the $a(\vec{z}, \vec{v})$ are precisely similar to those were proved in Theorem (3.2.1). So, it remains to show that the RHS of (69) is bounded. Now, by taking the absolute value for it and then using the inequality of CS to get

$$|\bar{F}(\vec{v})| \le |F(\vec{v})| + |a(\vec{w}, \vec{v})| + \left|(g_{1N}, v_1)_{\Gamma_N}\right| + \left|(g_{2N}, v_2)_{\Gamma_N}\right|.$$

By using (21) and (22) for the first two terms in RHS and the CS inequality for the other two terms to accrue.

$$|\bar{F}(\vec{v})| \le \gamma \|\vec{v}\|_{\left(H_0^1(\Omega)\right)^2} + \rho \|\vec{w}\|_{\left(H_0^1(\Omega)\right)^2} \|\vec{v}\|_{\left(H_0^1(\Omega)\right)^2} + \|g_{1N}\|_{L^2(\Gamma_N)} \|v_l\|_{L^2(\Gamma_N)} + \|g_{2N}\|_{L^2(\Gamma_N)} \|v_l\|_{L^2(\Gamma_N)}$$

Since $\|\vec{w}\|_{\left(H_0^1(\Omega)\right)^2} \le \lambda$, $\|g_{lN}\|_{L^2(\Gamma_N)} \le \bar{\sigma}_l, \bar{\sigma} = \max(\bar{\sigma}_l)$, and from the Theorem2.5, the above inequality gives

$$|H(\vec{v})| \le \bar{\gamma} \|\vec{v}\|_{\left(H_0^1(\Omega)\right)^2}, \text{ where } \bar{\gamma} = \gamma + \rho\lambda + 2\bar{\sigma}. \quad (70)$$

Conclusions

The LMTH was employed successfully to prove the existence of a unique COS (in infinite dimension) of the WF for the CEPDEs with each of the four different types of the BCs (DBCs, NBCs, RBCs and MBCs). Also, the PFI was used successfully in the proof of the coercivity of the bilinear form where the BCS was homogeneous, while it was not applicable fall to applied in case that the BCS became nonhomogeneous, and to avoid this problem the generalized PFI was used instead of it successfully, beside the TRO. The result of this paper give us and many other investigators the green light to solve these problems numerically.

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