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Passer Journal of Basic and Applied Sciences

Passer 7 (Issue 2) (2025) 729-732

Original Article

On the Maximum Diameter of Graphs

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Received 12 January 2025; revised 20 July 2025;
accepted 09 February 2025; available online 03 April 2025

DOI: 10.24271/psr.2025.499398.1897

ABSTRACT

In this work, we studied the concept of maximum diameter for a few graph kinds (complete graph, hypercube graph, and generalized Petersen graph). We calculate the maximum diameter values denoted by $f(tt, D)$ of an altered graph G with n $(n \geq 4)$ vertices, which is obtained by eliminating tt edges from a complete graph K_n that has n vertices and contains the greatest quantity of edges in all simple graphs.

Keywords: complete graph, hypercube graph, generalized Petersen graph, edges deletion problem, maximum diameter.

1 Introduction

Graph theory has experienced a surge in popularity in recent years due to its uses in a number of areas in mathematics and science, such as biology, engineering, and social science [1]. The network's mathematical representation is represented by a graph. In most research, a network typically refers to an actual system, whereas a graph typically refers to a mathematical model [2]. Fault tolerance is an important concept in communication network design since it measures the longest message delay experienced by a message traveling between any two vertices inside the network [3]. A crucial consideration in network architecture is communication. As a result, it is challenging to disrupt a strong network, and it ought to continue sending and receiving data even in the event that nodes or links malfunction [4]. Graph theory approaches can be used to investigate a network's dependability and effectiveness, and the network's connectivity can be used to assess its dependability. Network designers generally favor graphs that can withstand strong shocks when errors arise on the nodes or links [5]. The impact of connection failures in networks was studied using diameter, which is a quantification of network efficiency. It is useful to know how removing edges and vertices from a graph changes it, because a network represented by these graphs may become less efficient. A graph's diameter can be changed by removing or adding edges [6].

Numerous authors have investigated how removing edges affects the diameter of a graph. For instance, Harary and Graham studied the effect of decreasing and increasing edges on the diameter of certain hypercube graphs [7]. Bouabdallah et al. discussed changing the diameter of a hypercube graph after removing the maximum edges from a hypercube graph [8]. If d is an even or odd (greater than 2) diameter, Schoone and others derived a lower bound for the maximum diameter given by $(d-1)t + d$ or $t(d-2)+d+2$ respectively. They found an upper bound for $(td+d)$. Also, they take the complete graph as the particular case [9]. The maximum diameter $f(t,d)$ of the connected graph G, which is produced by deleting t edges, was examined by Najim and Jun-Ming. They demonstrated that $f(t,d) \geq d + (d-1)t + 1$ were $7+t \geq d \geq 4+t$ and $t > 3, d = 2m + t(2m-1)$ and $m \geq 1$, as well as $t = 4$ and $d = 1 + 10m$[10]. After removing certain edges from a hypercube graph while maintaining the graph's connectivity, Jasim and Najim obtained the exact values of the maximum diameter of an altered graph [11]. Jasim and Najim showed the values of the maximum diameter of the graph $GP(n,k,t)$ that was achieved by deleting t edges from $GP(n,k)$ with $k = 1, n \geq 3$ (where $GP(n,k)$ denotes the generalized Petersen graph) [12]. After a certain edge deletion from a complete graph, Jasim and Najim investigated how the diameter increased in a complete graph. They indeed showed the exact values of $f(t,d)$ for every connected graph G_i with n vertices and $i \geq 1$, achieved by deleting t edges from a complete graph [13]. In this paper, we give a general result by giving a general formula for calculating the maximum diameter for a graph that is acquired by eliminating specific edges from any graph G with n vertices.

Let G be undirected and simple graph, containing $V(G)$ as the set of its vertices and $E(G)$ as the set of its edges. Assume that $d_G(u_1,u_2)$ refers to the distance between the vertices u_1 and u_2, then the greatest $d_G(u_1,u_2)$ in G is known as its diameter,

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Peer-reviewed under the responsibility of the University of Garmian.

denoted by D, such that $D = max(d_G(u_1, u_2))$ and $u_1, u_2 \in V(G)$[11]. Assume that $\{e\}$ is a set of edges in G. Then the altered graph G' results by deleting $\{e\}$ from G and $G - \{e\} = G'$. It is clear that the subgraph G' has the set of edges $E(G - \{e\}) = E(G')$ and the set of vertices $V(G) = V(G')$[14].

2 Edges Deletion Problem of Graphs

Within this part, we look into some specific graph types and their properties and talk about how the diameter will increase as we remove a certain number of edges from graphs.

2. 1 Edges Deletion Problem of $Gp(n, k)$

If k and n are integers, $1 \leqslant k \leqslant \lfloor\frac{(n-1)}{2}\rfloor$ (with k is the skip) and $n \geq 3$, then the graph $Gp(n, k)$ is undirected graph whose order and size are $2n$ and $3n$, respectively. $Gp(n, k)$ is a 3-regular graph containing $V(Gp)$ (as its set of vertices) which is defined by $V(Gp) = \{u_i, v_i, 1 \leqslant i \leqslant n\}$, and a set of edges denoted by $E(Gp)$ that are defined by $E(Gp) = \{u_i u_i + 1, u_i v_i, v_i v_i + k, 1 \leqslant i \leqslant n\}$, where i is an integer (all the indices larger than n we make it modulo n). The edges mentioned in $E(Gp)$ are called inner edges (v_i, v_{i+k}), spoke edges (u_i, v_i), and outer edges (u_i, u_{i+1}) [15,16]. It can be seen that when $n = 2k$, the resulting graph is not cubic; therefore, it must be $n = 2k$. In $Gp(n, k)$, there exists an isomorphism $Gp(n, k) = Gp(n, n - k)$, as well as one or more inner cycles and one outer cycle [17]. In Lemma 1, Jasim and Najim determined the general formula for the maximum diameter $f(n, t)$ of $Gp(n, k, t)$ achieved by t edges deleted from $Gp(n, k)$ with $k = 1$ and $n \geq 3$ [12].

Lemma 1. If $n \geq 3$ and $2 \leqslant t \leqslant n + 1$, then:

$$f(n, t) = n + t - 2. \tag{1}$$

2. 2 Edges Deletion Problem of Q_n

The hypercube graph $Q_n, n \geq 3$ is an undirected graph of dimension n consisting of $n2^{n-1}$ edges and 2^n vertices. The representation of each vertex by an n −length binary string, so that every two vertices adjacent to their labels differ by exactly one bit [18]. Jasim and Najim are determined the general formula for the maximum diameter $f(t)$ of an altered graph obtained after removing t edges from $Q_n, n \geq 3$, keeping the output graph connected [11]. In Lemma 2, they calculated the general formula for the most edges that can be eliminated from Q_n with $n \geq 3$ and the general formula for the maximum diameter that is associated with deletion [11].

Lemma 2. If $n \geq 3$, then:

$$f(t) = 2^n - 1 \text{ and } t = n2^{n-1} - (2^n - 1). \tag{2}$$

2. 3 Edges Deletion Problem of K_n

A complete graph K_n is an undirected and simple graph. Each pair of unique vertices in K_n are adjacent, the greatest quantity of edges in all simple graphs. Note that $K_n$has n vertices and $\frac{1}{2}n(n - 1)$ edges. A complete graph is $(n - 1)$ regular graph with diameter $d = 1$ [19]. Jasim and Najim determined the maximal diameter $f_n(tt, D)$ of altered connected graphs $G_i, i \geq 1$ obtained from a graph G after deleting tt edges. A graph G with n vertices obtained from a K_n after deleting t edges. The diameter of a graph G is represented by D, and $D \geq 2$. In Lemma 3, they calculated the general formula for the maximal diameter of altered connected graphs after deleting t and tt edges [13].

Lemma 3. If $tt \leqslant \frac{1}{2}(n - 1)(n - 2)$ and $D \geq 2$, then:

$$f(tt, D) = \lfloor n + \tfrac{1}{2} - \sqrt{\left(n - \tfrac{3}{2}\right)^2 - 2tt}\rfloor. \tag{3}$$

3 Main result

Within this part, we generalize Lemma 3, by giving a general formula for calculating the maximum diameter for a graph that is obtained by deleting certain edges from any graph G with n vertices.

Theorem 1. If $d_j \geq 2$ with $j \geq 1$ and $1 \leqslant tt \leqslant \frac{1}{2}(n - 1)(n - 2)$ with $n \geq 4$, then:

$$2 \leq f_{G_n}(tt, d_j) \leq \lfloor n + \tfrac{1}{2} - \sqrt{\left(n - \tfrac{3}{2}\right)^2 - 2tt}\rfloor. \tag{4}$$

Proof. A connected graph G_i $(i \geq 1)$ is gotten by removing t edges from K_n, with $n \geq 4$. Suppose that G_j $(j \geq 1)$ is a connected graph with $n \geq 4$ achieved by deleting tt edges from G_i , with $i \geq 1$. If d_j represents the maximum diameter of G_j $(j \geq 1)$, then: $d_j \geq 2$ since $G_j = (\mathrm{K}_n - \mathrm{t}) - \mathrm{tt}$, for all $(j \geq 1)$. From Lemma 3, we see that

$$f(tt, D) = \lfloor n + \tfrac{1}{2} - \sqrt{\left(n - \tfrac{3}{2}\right)^2 - 2tt}\rfloor,$$

$f(tt, D)$ represented the maximum of d_j, with $\mathrm{j} \geq 1$. The most quantity of edges that can be deleted from G_j $(j \geq 1)$ is given by $tt \leqslant \frac{1}{2}(n - 1)(n - 2)$. Then we can easily get:

$$2 \leq f_{Gn}(tt, d_j) \leq \lfloor n + \tfrac{1}{2} - \sqrt{\left(n - \tfrac{3}{2}\right)^2 - 2tt}\rfloor,$$
$$\text{for } 1 \leq tt \leqslant \tfrac{1}{2}(n - 1)(n - 2).$$

$f_{Gn}(tt, d_j)$ represents the maximum diameter of G_j $(j \geq 1)$ with order n. □

The following examples demonstrate the apply Theorem 1, to determine the maximum diameter of the modified graph that results from the removal of a certain edges from the original graph

4 Illustrative examples

As applications for Theorem 1, we consider certain types of altered graphs, obtained after deleting t edges from complete graph K_n for some n. The studied altered graphs are given as follows:

4. 1 The hypercube graph $\boldsymbol{Q_3}$

The hypercube graph Q_3 is clearly obtained from the complete graph K_8. More precisely, after deleting t edges ($t = 16$) from K_8, we get Q_3 as shown in "Figure 1,". In "Figure 2," illustrates the process of edges deletion from Q_3 and calculates the maximum diameter after deletion.

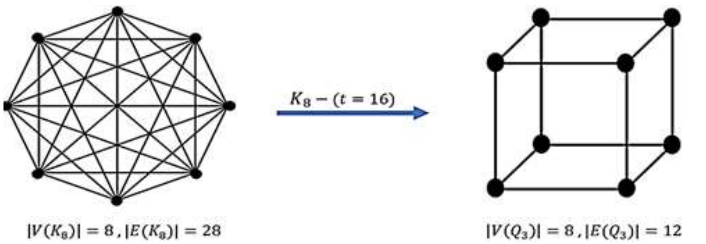

Figure 1: Deriving Q_3 from K_8.

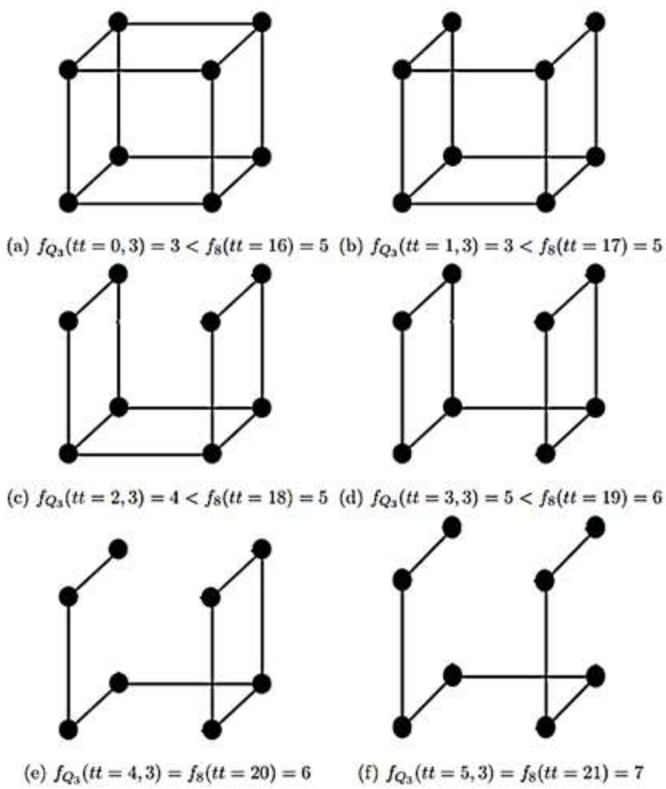

Figure 2: The process of edges deletion from Q_3.

4. 2 The Generalized Petersen graph $\boldsymbol{GP(5, 1)}$

The Generalized Petersen graph $GP(5,1)$ is clearly obtained from the complete graph K_{10}. More precisely, after deleting t edges ($t = 30$) from K_{10}, we get $GP(5,1)$ as shown in Figure 3. In Figure 4, illustrates the process of edges deletion from $GP(5,1)$ and calculates the maximum diameter after deletion.

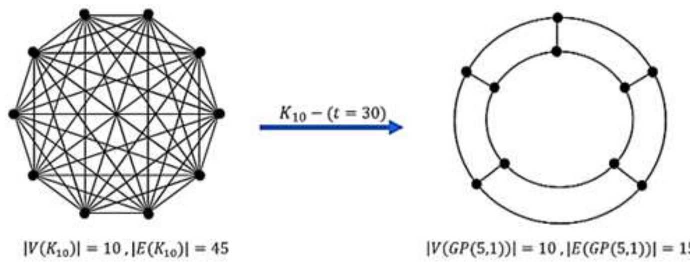

Figure 3: Deriving GP(5,1) from K_10.

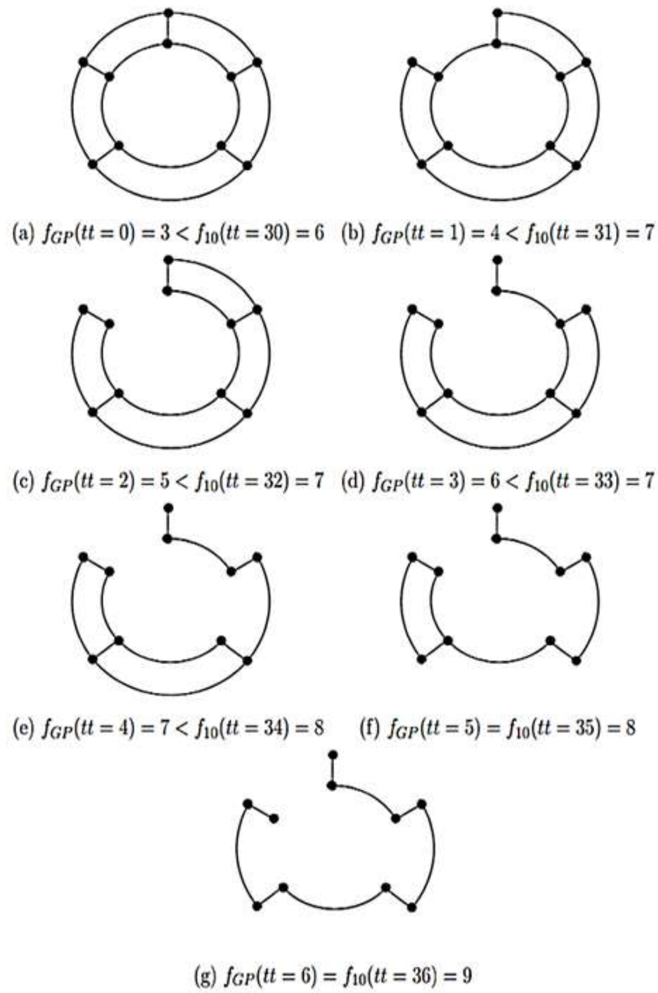

Figure 4: The process of edges deletion from GP (5,1).

As seen in the earlier instances, Theorem 1 may be used to determine the maximum diameter of any kind of graph following the removal of specific edges.

5 Conclusion

The results of this paper provided a general formula for finding the maximum diameter of the graph G of order $n \geq 4$, G is obtained by deleting t edges from the complete graph K_n, with $n \geq 4$.

Conflict of interests

The authors declare that they have no conflict of interest.

Funding

The author received no financial support for the research, authorship, and publication of this article.

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