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Original Article

Generalized Prime Systems from a Topological Perspective

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ABSTRACT

This study focuses on developing a topological framework for studying a generalized prime system (GPS), based on constructing specialized topological spaces determined by such systems (GPS-determined topological spaces). The research demonstrates how nested open sets and discrete thresholds govern the distribution and asymptotic behavior of generalized primes. The main contributions include a complete topological characterization of generalized prime counting functions, systematic methods for constructing generalized primes, and a concrete example illustrating the practical application of these theoretical results. This work enriches traditional analytical approaches by introducing new topological tools, opening promising interdisciplinary research avenues in analytic number theory and topology.

Keywords: Topological characterization, generalized primes, generalized prime system, GPS-determined topological spaces.

1 Introduction

The distribution of prime numbers is a central theme in analytic number theory, having received extensive attention since the 19th century due to its deep connections with analytic functions**,** particularly the Riemann zeta function. In the 1930s, Arne Beurling [1] extended these ideas by introducing generalized prime systems. While most previous studies have relied on analytic and number-theoretic approaches, this research offers a new perspective by employing topological methods in the study of such systems. It develops a topological framework that contributes to establishing new links between number theory and topology.

By constructing specialized topological spaces determined by GPS, this work demonstrates that nested open sets and discrete thresholds play a fundamental role in governing the distribution and asymptotic behavior of generalized prime numbers. The core contributions of this study are: (1) a complete topological characterization of generalized prime counting functions, (2) systematic methods for constructing generalized primes, and (3) a concrete example is presented to illustrate the practical relevance of these theoretical insights.

These findings provide topological tools that enrich the traditional analytic study of generalized primes and open promising avenues for interdisciplinary research. The structure of this work is as follows: Chapter Two introduces the fundamental concepts; Chapter Three defines the GPS-determined sets and their associated topologies; Chapter Four presents the main results; and Chapter Five presents the conclusion.

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2 Fundamental Concepts

Before exploring the topological framework, we first review the key analytical tools underlying generalized prime systems. For further elaboration, the reader is referred to [1], [2], [3], [4], and [10].

2. 1 Notation and Preliminaries

Let B be the space of all right-continuous functions $q:\mathbb{R} \to \mathbb{C}$ that are zero on $(-\infty, 1)$ and of local bounded variation (see [2]). Let $B^+ \subset B$ denote the subspace consisting of all increasing functions. For any real number a, we define

- $B_a = \{q \in B: q(1) = a\}$,
- $B_a^+ = B^+ \cap B_a$.

The Mellin-Stieltjes convolution of q, $w \in B$ is defined by

$$(q*w)(t) = \int_{1-}^{t} q\left(\frac{t}{x}\right) dw(x).$$

If $q \in B_1$, then there exists a function $w \in B_0$ such that

$$q = \exp * w,$$

which means

$q = \sum_{n=0}^{\infty} \frac{w^{*n}}{n!}$,

where $w^{*n} = w * w^{*(n-1)}, w^{*0} = i$. Also, $q = \exp * w \leftrightarrow q * w_L = q_L$, where $q_L \in B$ is defined by

$q_L = \int_1^t \log x \, dq(x)$, for $t \geq 1$. (see [3])

For $q \in B$, the Mellin transform of q is given by

$\hat{q}(s) = \int_{1-}^{\infty} \frac{1}{t^s} dq(t)$,

which exists if $q(t) = O(t^D)$ for some $D \geq 0$. Note that

$\widehat{q * w} = \hat{q}\,\hat{w}$ and $\widehat{\exp * q} = \exp \hat{q}$.

2. 2 Generalized Prime System (GPS)

The structure of this system is defined by an increasing sequence of positive real numbers

$\dot{P} = \{b_1, b_2, b_3, \dots\}$,

satisfying the following conditions:

- $b_1 > 1$,
- $b_1 \leq b_2 \leq \cdots \leq b_i \leq \cdots$,
- $\lim\limits_{i \to \infty} b_i = \infty$.

The elements b_i are called generalized primes. From these, one can form the associated generalized integers (denoted by $\ddot{N}$); that is, the numbers of the form

$b_1^{a_1} b_2^{a_2} \dots b_r^{a_r}$, where $r \in \mathbb{N}$ and $a_1, \dots, a_r \in \mathbb{N}_0$.

Each generalized prime system is associated with two fundamental counting functions:

- The generalized prime counting function, denoted by $\dot{\pi}_p$, is defined as

$\dot{\pi}_p(t) = \sum_{\substack{b \leq t \\ b \in \dot{P}}} 1$,

- The generalized integer counting function, denoted by $\dot{N}_p$, is defined as

$\dot{N}_p(t) = \sum_{\substack{m \leq t \\ m \in \ddot{N}}} 1$.

These systems are classified as discrete systems, where their associated counting functions $\dot{\pi}_p$ and $\dot{N}_p$ are step functions with integer jumps.

The weighted generalized prime counting function $\dot{\Pi}_p$ is defined as

$\dot{\Pi}_p(t) = \sum_{n=1}^{\infty} \frac{1}{n} \dot{\pi}_p \left(t^{\frac{1}{n}}\right)$.

This function is related to the generalized Chebyshev function $\dot{\psi}_p$ via the following relation

$\dot{\psi}_p(t) = \int_1^t \log u \, d\dot{\Pi}_p(u)$.

The generalized (or Beurling) zeta function $\dot{\zeta}_p$, associated with a generalized prime system, is given by

$\dot{\zeta}_p(s) = \int_{1-}^{\infty} \frac{1}{t^s} d\dot{N}_p(t) = \sum_{m \in \ddot{N}} m^{-s}$.

It is related to $\dot{\Pi}_p$ via

$\dot{\zeta}_p(s) = \exp\{\int_1^{\infty} t^{-s} \, d\dot{\Pi}_p(t)\}$,

for all $s \in \mathbb{C}$ where the series $\sum_{m \in \ddot{N}} m^{-s}$ converges.

The concept of "generalized primes", as defined earlier, can be extended to include more general increasing functions, (not necessarily step functions). This leads to the following definitions. (see [1], [4], [10])

2. 2. 1 Definition

An outer generalized prime system (OGPS) is a pair $(\dot{\Pi}, \dot{N})$ with $\dot{\Pi} \in B_0^+$, $\dot{N} \in B_1^+$ such that

$\dot{N} = \exp * \dot{\Pi}$.

2. 2. 2 Definition

A generalized prime system is an OGPS for which there exists $\dot{\pi} \in B_0^+$ such that

$\dot{\Pi}(t) = \sum_{n=1}^{\infty} \frac{1}{n} \dot{\pi}\left(t^{\frac{1}{n}}\right)$. ([4] Def. 1.3)

2. 2. 3 Remark

a) Using Möbius inversion, $\dot{\pi}(t)$ can be expressed as

$\dot{\pi}(t) = \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \dot{\Pi}\left(t^{\frac{1}{w}}\right)$.

Since $\dot{\Pi}\left(t^{\frac{1}{w}}\right)$ decreases with w and $\sum_{w=1}^{\infty} \frac{\mu(w)}{w}$ converges**,** this sum always converges for $\dot{\Pi} \in B^+$.

b) A generalized prime system is discrete if $\dot{\pi}$ is a step function with integer jumps. In this case, the step is the multiplicity, and the generalized primes are the discontinuities of $\dot{\pi}$. ([4], p.5)

c) An OGPS $(\dot{\Pi}, \dot{N})$ is continuous if $\dot{N}$ (and hence, $\dot{\Pi}$) is continuous in $(1,\infty)$.

d) For an OGPS $(\dot{\Pi}, \dot{N})$, let $\dot{\psi} = \dot{\Pi}_L$ denote the Chebyshev function. Note that $\dot{\psi} \in B_0^+$ and that

$\dot{N}_L = \dot{\psi} * \dot{N}$ is equivalent to $\dot{N} = \exp * \dot{\Pi}$. (see [2], [3])

3 GPS-Determined Sets and the Associated Topology

We define a collection of subsets of $Y = [1, \infty)$, called GPS-determined sets, as follows:

3. 1 Definition

For a function $f \in B$, the set Y_n^f is defined by

$Y_n^f := \{t \in Y \mid f(t) \geq n\}$,

for all $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$. The set Y_n^f is called a GPS-determined set in Y if the function f satisfies the following conditions:

1. $f(1) = 0$, and $f(t) \to \infty$ as $t \to \infty$.

2. f is continuous and strictly increasing on Y.

3. f is linked to an outer generalized prime system $(\dot{\Pi}, \dot{N})$ via the formula:

$f(t) = \sum_{w=1}^{\infty} \frac{\mu(w)}{w} \dot{\Pi}\left(t^{\frac{1}{w}}\right)$,

where μ is the Möbius function.

3. 2 Proposition

Let Y_i^f and Y_j^f be GPS-determined sets in $[1, \infty)$. Then:

a. $Y_i^f = Y$ whenever $i = 0$.

b. $Y_i^f \cap Y_j^f = Y_{\max(i,j)}^f$.

c. $Y_i^f \cup Y_j^f = Y_{\min(i,j)}^f$.

d. The complement of Y_i^f in Y is $Y \setminus Y_i^f = \{t \in Y \mid t < i\}$.

e. $Y_i^f \subseteq Y_j^f$ whenever $j \leq i$.

Proof. Straightforward

3. 3 Proposition

For any $n \in \mathbb{N}_0$, the GPS-determined set Y_n^f is non-empty.

Proof. Straightforward

3. 4 Definition

The GPS-determined topology T_{GPS} on Y is defined as

$T_{GPS} := \{Y_n^f \subseteq Y \mid n \in \mathbb{N}_0\} \cup \{\emptyset\}$,

where Y_n^f is a GPS-determined set in Y, and $n \in \mathbb{N}_0$. The pair (Y, T_{GPS}) is referred to as a GPS-determined topological space. It is straightforward to verify that T_{GPS} satisfies the axioms of a topology. (See 3. 5 Theorem).

3. 5 Theorem

T_{GPS} is a topology on Y.

Proof.

We verify that T_{GPS} satisfies the three conditions for a topology:

i) Since $Y = Y_0^f$, it follows that Y belongs to T_{GPS} as required. The empty set $\emptyset$ also belongs to T_{GPS} (by definition).

ii) Let $U_1, U_2 \in T_{GPS}$. For each $m \in \{1,2\}$, U_m is either a GPS-determined set in Y or $U_m = \emptyset$. Let $S = U_1 \cap U_2$. We must prove that $S \in T_{GPS}$; that is, we must prove that S is a GPS-determined set in Y, or that $S = \emptyset$.

If $U_1 = \emptyset$ or $U_2 = \emptyset$, then $S = U_1 \cap U_2 = \emptyset$.

If both $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$, then there exist $Y_i^f \subseteq Y$ and $Y_j^f \subseteq Y$ such that $U_1 = Y_i^f$ and $U_2 = Y_j^f$. Therefore,

$U_1 \cap U_2 = Y_i^f \cap Y_j^f = Y_{\max(i,j)}^f$,

which implies $S = Y_{\max(i,j)}^f$, as desired. Thus, in either case, $S \in T_{GPS}$.

iii) The proof of condition (iii) follows a similar approach to that of condition (ii).

Hence, (Y, T_{GPS}) is a GPS-determined topological space.

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4 Asymptotic Behavior and Applications

4. 1 Theorem

Let $Y = [1, \infty)$, and let $T_{GPS} = \{Y_n^f \subseteq Y \mid n \in \mathbb{N}_0\} \cup \{\emptyset\}$ be a GPS-determined topology on Y. Then, there exists a sequence of generalized primes $\dot{P} = \{b_n\}_{n\geq 1}$ directly derived from T_{GPS}, such that:

1- For any $t \geq 1$ (excluding the trivial open set $Y_0^f = Y$):

$\dot{\pi}_p(t) =$ Number of open sets Y_n^f containing t.

2- The function $\dot{\pi}_p$ satisfies:

$\dot{\pi}_p(t) = f(t) + O(1)$.

Proof.

The topology T_{GPS} is generated by the GPS-determined sets

$Y_n^f = \{ t \in Y \mid f(t) \geq n\}$, $n \in \mathbb{N}_0$.

These sets satisfy the nested property

$Y_{n+1}^f \subset Y_n^f$ for all $n \in \mathbb{N}_0$,

since $f(t) \geq n + 1 \Longrightarrow f(t) \geq n$.

This nested structure imposes a rigid order on Y, creating a sequence of "thresholds" that encode the asymptotic behavior of f.

Note that $Y_0^f = Y$ is the trivial open set, as $f(t) \geq 0$ holds for all $t \in Y$. We exclude Y_0^f from our analysis because it does not provide meaningful information about the structure of $\dot{P}$.

For each $n \in \mathbb{N}$, define the threshold M_n:

$M_n = \min\{ t \in Y \mid f(t) \geq n\}$.

By definition, M_n is the smallest element in the open set Y_n^f, i.e.,

$M_n = \min(Y_n^f)$.

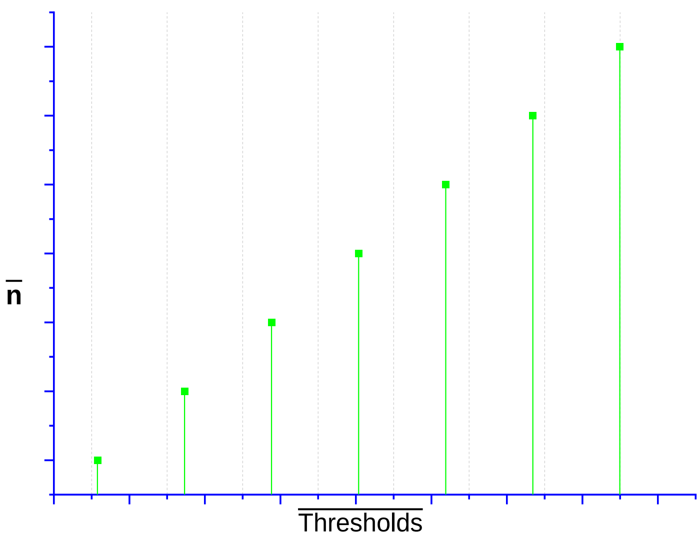
Since f is strictly increasing, continuous, and unbounded ($f(t) \to \infty$ as $t \to \infty$), each $Y_n^f \neq \emptyset$ and can be written as

$Y_n^f = [M_n, \infty)$.

The sequence $\{M_n\}$ satisfies

$1 < M_1 < M_2 <$ ... and $M_n \to \infty$ as $n \to \infty$.

The following figure illustrates the distribution of thresholds M_n. The diagram highlights how these thresholds increase sequentially with n, reflecting the nested structure of the open sets Y_n^f. These thresholds form the foundation for constructing $\dot{P}$.



We construct the sequence of generalized primes $\dot{P} = \{b_n\}$ by setting,

$b_n = M_n$.

The properties of $\{M_n\}$ ensure that

$1 < b_1 < b_2 <$ ... and $b_n \to \infty$) as $n \to \infty$.

The associated counting function $\dot{\pi}_p$ is defined by

$\dot{\pi}_p(t) = \sum_{\substack{b_n \leq t \\ n \in \mathbb{N}}} 1$.

Since $M_n = b_n$,

$\dot{\pi}_p(t) = \max\{n \in \mathbb{N} \mid M_n \leq t\}$.

By definition,

$Y_n^f = [M_n, \infty)$.

Therefore, for a given $t \geq 1$,

$t \in Y_n^f$ if and only if $M_n \leq t$.

Thus, the number of open sets Y_n^f containing t (excluding the trivial open set $Y_0^f = Y$) is equal to the number of indices n for which $M_n \leq t$. This is precisely the definition of $\dot{\pi}_p$:

$\dot{\pi}_p(t) = \sum_{\substack{b_n \leq t \\ n \in \mathbb{N}}} 1$.

Thus, we have

$\dot{\pi}_p(t) =$ Number of open sets Y_n^f containing t.

For any $t \geq 1$, if there exists $n \in \mathbb{N}$ such that $M_n \leq t < M_{n+1}$, then

$\dot{\pi}_p(t) = n$.

Since f is strictly increasing and continuous, we have

$n \leq f(t) < n + 1$ for $t \in [M_n, M_{n+1})$,

which implies

$0 \leq f(t) - \dot{\pi}_p(t) < 1$. If $t < M_1$, then $\dot{\pi}_p(t) = 0$, and since $f(t) < 1$ for $t < M_1$, we have

$0 \leq f(t) - \dot{\pi}_p(t) < 1$.

Thus, in all cases, we conclude

$\dot{\pi}_p(t) = f(t) + O(1)$ for all $t \geq 1$.

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4. 2 Example

Let $T_{GPS} = \{Y_n^f \subseteq Y \mid n \in \mathbb{N}_0\} \cup \{\emptyset\}$ be an GPS-determined topology on $[1, \infty)$, where f is defined as

$f(t) = \sum_{r=1}^{\infty} \frac{(\log t)^r}{r!\, r\, \zeta(r+1)}$,

here, ζ denotes the Riemann zeta function.

Then, there exists a discrete generalized prime system $\dot{P}$ with the following properties:

1- The function $\dot{\Pi}_p$ satisfies

$\dot{\Pi}_p(t) = \int_1^t \frac{1-u^{-1}}{\log u} du + O(\log\log t)$.

2- For some positive constant $\rho > 0$ and for all $\varepsilon > 0$, the function $\dot{N}_P$ satisfies

$$\dot{N}_P(t) = \rho t + O(t\, e^{-\frac{\sqrt{1-\varepsilon}}{4}(\log t \log\log t)^{\frac{1}{2}}}).$$

This asymptotic behavior implies that the generalized zeta function $\dot{\zeta}_p$ satisfies

$$\dot{\zeta}_p(\theta + ix) \ll (x^q), \text{ as } x \to \infty,$$

in this region:

$$1 - \theta \ll \frac{1}{4}\sqrt{\frac{(1-\varepsilon)\log x}{x}},$$

for some $q > 0$.

Proof.

We apply Theorem 4.1 with the function

$$f(t) = \sum_{r=1}^{\infty} \frac{(\log t)^r}{r!\, r\, \zeta(r+1)}.$$

This allows us to construct a sequence of generalized primes P $= \{b_n\}_{n\geq 1}$, where

$$1 < b_1 < b_2 < \cdots,$$

and the function $\dot{\pi}_p$ satisfies

$$\dot{\pi}_p(t) = \sum_{\substack{M_n \leq t \\ n\in\mathbb{N}}} 1$$

$$= f(t) + O(1).$$

Here,

$$M_n = \min\{\, t \in Y \mid \sum_{r=1}^{\infty} \frac{(\log t)^r}{r!\, r\, \zeta(r+1)} \geq n\}$$

is the smallest element in the open set Y_n^f, and $M_n = b_n$.

This approximation $\dot{\pi}_p(t) = f(t) + O(1)$ is crucial for deriving the asymptotic behavior of $\dot{\Pi}_p$ and $\dot{N}_P$. By definition,

$$\dot{\Pi}_p(t) := \sum_{n=1}^{\infty} \frac{1}{n}\, \dot{\pi}_p\left(t^{\frac{1}{n}}\right) = \sum_{n\leq \log t/\log M_1} \frac{1}{n}\, \dot{\pi}_p\left(t^{\frac{1}{n}}\right),$$

where the terms become identically zero as soon as $n > \log t/\log M_1$.

Thus,

$$\dot{\Pi}_p(t) = \sum_{n\leq \log t/\log M_1} \frac{1}{n}\left(f\left(t^{\frac{1}{n}}\right) + O(1)\right)$$

$$= \sum_{n\leq \log t/\log M_1} \frac{1}{n} f\left(t^{\frac{1}{n}}\right) + O\left(\sum_{n\leq \log t/\log M_1} \frac{1}{n}\right)$$

$$= \sum_{n\leq \log t/\log M_1} \frac{1}{n} f\left(t^{\frac{1}{n}}\right) + O(\log(\log t/\log M_1)).$$

This leads to

$$\dot{\Pi}_p(t) = \sum_{n=1}^{\infty} \frac{1}{n} f\left(t^{\frac{1}{n}}\right) + O\left(\sup_{c\in[1,M_1]} (f(c))' \sum_{n>\log t/\log M_1} \frac{t^{\frac{1}{n}-1}}{n} + \log\log t\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} f\left(t^{\frac{1}{n}}\right) + O(\log\log t)$$

$$= \dot{\Pi}_{\overline{p0}}(t) + O(\log\log t),$$

where we have set

$$\dot{\Pi}_{\overline{p0}}(t) := \sum_{n=1}^{\infty} \frac{1}{n} f\left(t^{\frac{1}{n}}\right) = \int_1^t \frac{1-u^{-1}}{\log u}\, du.$$

Since $\dot{\psi}_{\mathbf{p}}(t) := \int_1^t \log u\, d\dot{\Pi}_p(u)$, we first compute

$$\int_1^t \log u\, d\dot{\Pi}_{\overline{p0}}(t) = t - \log t - 1.$$

This gives

$$\dot{\psi}_{\mathbf{p}}(t) = \int_1^t \log u\, d\dot{\Pi}_p(u) = \int_1^t \log u\, d\dot{\Pi}_{\overline{p0}}(u)) + \int_1^t \log u\, d(\dot{\Pi}_p(u) - \dot{\Pi}_{\overline{p0}}(u))$$

$$= t - \log t - 1 + (\dot{\Pi}_p(t) - \dot{\Pi}_{\overline{p0}}(t))\log t - \int_1^t \frac{(\dot{\Pi}_p(u) - \dot{\Pi}_{\overline{p0}}(u))}{u}\, du$$

$$= t + O(\log t \log\log t) + O\left(\int_1^t \frac{(\log\log u)}{u}\, du\right)$$

$$= t + O(\log t \log\log t)$$

Since $\lim_{t\to\infty} \frac{\log t \log\log t}{t^{\varepsilon}} = 0$ for any $\varepsilon > 0$, it follows that

$$\dot{\psi}_{\mathbf{p}}(t) = t + O(t^{\varepsilon}),\ \forall \varepsilon > 0.$$

Applying [2. 2 Theorem in [7]], we derive the asymptotic formula for $\dot{N}_P(t)$:

$$\dot{N}_P(t) = \rho t + O(t\, e^{-\frac{\sqrt{1-\varepsilon}}{4}(\log t \log\log t)^{\frac{1}{2}}}),$$

for some positive constant $\rho > 0$.

Finally, we apply [2. 2 Theorem in [9]] with the function

$$k(t) = \frac{\sqrt{1-\varepsilon}}{4}(\log t \log\log t)^{\frac{1}{2}}.$$

Substituting t with e^t, we obtain

$$k(e^t) = \frac{\sqrt{1-\varepsilon}}{4}(\log e^t\ \log\log e^t)^{\frac{1}{2}} = \frac{\sqrt{1-\varepsilon}}{4}(t \log t)^{\frac{1}{2}}.$$

This implies

$$\frac{k(e^t)}{t} = \frac{1}{4}\sqrt{\frac{(1-\varepsilon)\log t}{t}}.$$

The function $\frac{k(e^t)}{t}$ is decreasing. Since all the conditions of the theorem are satisfied, we conclude that

$\dot{\zeta}_p(\theta + ix) \ll (x^q)$,

in this region:

for $1 - \theta \ll \frac{1}{4}\sqrt{\frac{(1-\varepsilon)\,\mathrm{logx}}{x}}$,

for some $q > 0$.

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5 Conclusion

Our main result (4. 1 Theorem) proves that the nested open sets $Y_n^f = [M_n, \infty)$, where $M_n = \min\{t \in Y \mid f(t) \geq n\}$, define the generalized prime sequence $\dot{P} = \{M_n\}$ and encode the asymptotic distribution of generalized primes via the relation $\dot{\pi}_p(t) = f(t) + O(1)$. 4. 2 Example provides an explicit implementation. These results yield: (i) a complete topological characterization of generalized prime counting functions, and (ii) systematic methods for constructing generalized primes. These results also open promising avenues for future interdisciplinary research at the intersection of number theory and topology.

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