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Original Article

Orthogonal Zigzag Operation in An Absolute Plane and Criteria for Quasi-Parallelism

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ABSTRACT

In works by Karzel, Pianta, Rostamzadeh, and Taherian, quasi-parallel lines — pairs of lines without intersection and common perpendicular— were introduced to classify absolute planes. However, the properties and characterization of this geometric relationship remain unexplored. In this paper, we answer the fundamental question, "When are two distinct lines in an absolute plane quasi-parallel?" Even more generally, "How to recognize two lines from their geometric relationship?" To answer these questions, we introduce a novel geometric operation called the orthogonal zigzag operation, which provides new criteria for quasi-parallelism and other line relationships. Additionally, it offers a new perspective on the classification of absolute planes. For distinct lines A and B and a point $x \in A$, the orthogonal projection π_A maps x to its foot on B, and π_B maps this foot back to a point $x'' \in A$. The composition $\pi_{AB} = \pi_B \circ \pi_A : A \to A$ which maps x to x'' is called the orthogonal zigzag operation. We show this operation characterizes all possible line relationships between lines: π_{AB} is identity if and only if A and B are coperpendicular; constant if and only if A and B are orthogonal; has a contraction to a fixed point if and only if A and B intersect but are not perpendicular; contracts to a fixed point (extends from a fixed point) if and only if A and B have a coperpendicular and the plane has hyperbolic (elliptic) congruence, respectively; and finally, it is a fixed-point-free isotone permutation if and only if A and B are quasi-parallel.

Keywords: General Absolute Geometry, Quasi-Parallel Lines, Zigzag Operation, Non-Euclidean Geometry.

1 Introduction

In the 19th century, Bolyai, Gauss, and Lobachevsky created non-Euclidean geometry. This development transformed mathematicians' understanding of geometric structures and provided new approaches to examine fundamental geometric relationships. This essential work challenged Euclid's parallel postulate and demonstrated that there are other consistent geometric systems besides Euclidean geometry. In the 20th century, mathematicians such as Hilbert, Klein, and Beltrami built upon these early findings to define and study the properties of what became known as hyperbolic and elliptic geometries.

Hilbert's axiomatic method represented a significant advance in this area. He transformed plane geometry into what he called "natural geometry" [1] by ignoring the parallel axiom on purpose and basing it on five basic sets of axioms. this systematic approach inspired later mathematicians to develop different axiomatic bases for geometric systems. Helmut Karzel's comprehensive study of absolute planes, which serves as the theoretical basis for this paper [2], stands out among these contributions.

Over the past few decades, the study of absolute geometry has evolved significantly, with researchers employing a variety of complementary methods. Karzel and Marchi [3] established a framework for characterizing general absolute geometries using Lambert-Saccheri quadrangles. Rostamzadeh and Taherian [4] contributed to classification problems in absolute geometries by proposing new approaches to identifying geometric features. They have used triangles and Lambert-Saccheri quadrangles by employing angles and sides.

The concept of "quasi-parallelism" (two non-intersecting lines without a common perpendicular), which is central to this research, has become a rich area of study. Karzel, Pianta, Rostamzadeh, and Taherian [5] developed fundamental concepts about quasi-ends that can be used to describe absolute planes. This represents the most general classification for absolute planes discovered so far. Subsequently, they conducted essential work on quasi-parallelism in general absolute planes [6], which enhanced our understanding of how lines relate to each other in these geometric systems. A quasi-end is a group of lines where

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no two intersect or are coperpendicular. It is a generalization of Greenberg's idea of ideal points [7].

Greenberg's important work on Bolyai's parallel construction [7] established important connections between classical constructions and modern geometric theory. His later studies [8] provided new insights about hyperbolic constructions that have influenced contemporary research approaches.

Considerable attention has been devoted to the axiomatic foundations of absolute geometry in recent years. In ordered absolute geometry, Struve [9] made critical advances that took the theory in essential new directions. Augat and Hähl [10] introduced a new set of axioms for absolute plane geometry that is stated in terms of congruence and collinearity providing an alternative understanding to the classical approaches. Pambuccian's works have enhanced our knowledge of fundamental axiomatic approaches of absolute geometries. For example, his study of the elementary Archimedean axiom in absolute geometry [11] and of ubiquitous axioms [12] has helped us learn more about the basic requirements for geometric systems. From a historical perspective, his discussion of Euclid's axiom system [13] helps us understand why different traditional methods considered certain axioms important.

The Thomsen-Bachmann correspondence has been used to examine the connection between geometric structures and algebraic methods. The comprehensive study [14, 15] by Struve and Struve found strong links between metric geometry and group theory. It demonstrated how geometric structures can be understood algebraically and how algebraic structures can be interpreted through geometric features. This exchange has been beneficial in connecting areas of mathematical study that initially appear quite different. Also, recent graph-theoretic approaches have shown promise in analyzing geometric relationships [16,17].

Characterizing quasi-parallel lines, or more generally the relationship between two lines, in absolute planes is still a topic of ongoing research with many critical unanswered questions, despite these noteworthy advancements. A theoretical knowledge gap exists since the literature currently in publication offers few methods for explicitly identifying whether two different lines in an absolute plane are quasi-parallel, that is, two lines that do not intersect or have a common perpendicular.

In this paper, we address these fundamental questions: "When are two distinct lines in an absolute plane quasi-parallel?" More generally: "How can we recognize the geometric relationship between any two distinct lines?“ To answer these questions systematically, we introduce a novel geometric operation called the orthogonal zigzag operation, which provides new criteria for quasi-parallelism and, more generally, for the geometric relationship of any two distinct lines—that is, orthogonality, intersection, parallelism, and common perpendicularity relationships. Additionally, it offers a new perspective on the classification of absolute planes.

Let $\boldsymbol{A} = (\boldsymbol{E}, \boldsymbol{G}, \boldsymbol{\alpha}, \equiv)$ be a general absolute plane in terms of Karzel [2] in which $\boldsymbol{E}$ is the set of points, $\mathbf{G}$ the set of lines, $\boldsymbol{\alpha}$ stands for the order structure, and $\equiv$ is the congruence. The word general refers to no assumption is made about continuity.

We denote points by $a, b, c, \dots, \alpha, \beta, \gamma, \dots$ and lines by $A, B, C, \dots$. We show the unique line passing through two distinct points a and b by $\overline{a,b}$. We recall the definition of the betweenness function $\boldsymbol{\alpha}$ [2]. Let $\boldsymbol{E}^{3'} = \{(x,y,z) | x \neq y \wedge z \in \overline{x,y}\}$, then:

$$\boldsymbol{\alpha}: \boldsymbol{E} \times \boldsymbol{E} \times \boldsymbol{E} \to \{-1,1\}; (a,b,c) \mapsto (a|b,c)$$

is called a betweenness function if it satisfies the following axioms:

- $(\boldsymbol{Z}\ \mathbf{1})$ $\forall a,b,c \in \boldsymbol{E}^{3'}, (a|b,c) = (a|c,b)$.
- $(\boldsymbol{Z}\ \mathbf{2})$ $\forall a,b,c,d \in \boldsymbol{E}\ with\ a \neq b,c,d$ we have $(a|b,c) = (a|b,d) \cdot (a|d,c)$.
- $(\boldsymbol{Z}\ \mathbf{3})$ For all distinct points a, b and c exactly one of $(a|b,c)$ or$(b|a,c)$, or $(c|a,b)$ is -1.
- $(\boldsymbol{ZP})$ For non-collinear points $a, b, c \in \boldsymbol{E}$, if x∈ $]b,c[\ and\ y \in]a,x[$ then $\overline{c,y} \cap]a,b[\neq \emptyset$.

$(a|b,c) = -1$ *when* a *is between* b *and* c, *and* $(a|b,c) = 1$ *otherwise.*

Let $]a,b[:= \{x \in \overline{a,b} \mid (x|a,b) = -1\}$ *be the open segment and* $[a,b] :=]a,b[\cup \{a,b\}$ *the closed segment determined by distinct points a and b. Also,* $\overrightarrow{a,b} = \{x \in \overline{a,b}\ |(a|x,b) = 1\}$ *and* $\overrightarrow{\overline{a,b}} = \{x \in \overline{a,b}\ |(a|x,b) = -1\}$ *are the half-lines determined by a and b such that* $\overline{a,b} = \overrightarrow{a,b} \uplus \{a\} \uplus \overrightarrow{\overline{a,b}}$. We denote two perpendicular lines A, $B \in \boldsymbol{G}$ *by* $A \perp B$, and if they are not perpendicular, we show it by $A \not\perp B$. Also, let $A^{\perp} := \{X \in \boldsymbol{G} \mid X \perp A\}$ be the set of all lines that are orthogonal to line A. An ordinary absolute plane can be defined by this property. Hence, an absolute plane is ordinary if for any two lines $A, B \in \boldsymbol{G}$ we have $A^{\perp} \neq B^{\perp}$**.** Also, we denote the unique line orthogonal to A from the point a by $(a \perp A)$.

A *Lambert-Saccheri quadrangle* (in short LS quadrangle) is the set of four distinct points $a, b, c, d \in E$ such that $\overline{d,a} \perp \overline{a,b} \perp \overline{b,c} \perp \overline{c,d}$. Let a' be the foot of the perpendicular line from a to $\overline{c,d}$ and LS be the set of all Lambert-Saccheri quadrangles. We have this classification: The set of rectangles LS_r if $a' = d$ (see Figure 1.a), the set of hyperbolic quadrangles LS_h *if* $a' \in]c,d[$(see Figure 1.b), and the set of elliptic quadrangles LS_e if $d \in]c,a'[$ (see Figure 1.c). If we set $\varepsilon = 0$ for $a' = d$ and $\varepsilon := (d|c,a')$ then $\varepsilon = 0$ characterizes the rectangles, $\varepsilon = 1$ the hyperbolic quadrangles, and $\varepsilon = -1$ the elliptic quadrangles. In [3] (2.3), it is proved that if $LS_h \neq \emptyset$ then $LS = LS_r$, if $LS_h \neq \emptyset$ then $LS = LS_h$ and if $LS_e \neq \emptyset$ then $LS = LS_e$. Consequently, we recall [3] that an absolute plane has *Euclidean congruence* if $LS_r \neq \emptyset$, *hyperbolic congruence* if $LS_h \neq \emptyset$, and *elliptic congruence* if $LS_e \neq \emptyset$. Absolute planes with Euclidean congruence are also called *singular* and all others are called *ordinary absolute planes*.

Figure 1: a. Euclidean congruence b. Hyperbolic congruence c. Elliptic congruence

For any two distinct lines $A, B \in \boldsymbol{G}$ in an absolute plane we have exactly one of the following situations (see Figure 2):

1. $A \cap B \neq \emptyset$ and $A \not\perp B$, i.e., A and B intersect but are not orthogonal.
2. $A \cap B \neq \emptyset$ and $A \perp B$, i.e., A and B are orthogonal.
3. $A \cap B = \emptyset$ *and* $A^{\perp} = B^{\perp}$, i.e., A and B are coperpendicular.
4. $A \cap B = \emptyset$ *and* $|A^{\perp} \cap B^{\perp}| = 1$, i.e., A and B exactly have one common perpendicular.
5. $A \cap B = \emptyset$ *and* $A^{\perp} \cap B^{\perp} = \emptyset$, i.e., A and B have no intersection and no coperpendicular. In this case, following [5], we call them quasi-parallel lines, and we denote $A \ddagger B$.

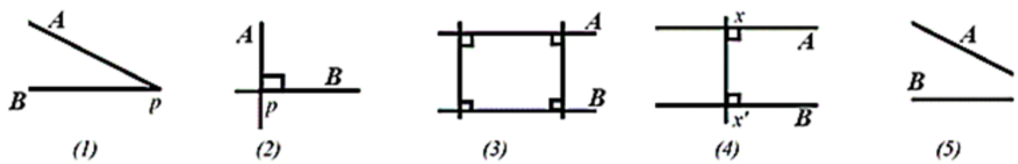

Figure 2: Any two distinct lines in an absolute plane have exactly one of these situations.

To characterize the relationship between mutual lines, especially quasi-parallel lines, in this paper we introduce the zigzag operation: Suppose that A and B are two distinct lines of an absolute plane and x is a point on A. Additionally, let $(x \perp B)$ be the unique line through point x and perpendicular to B. Now, $\pi_A: A \to B;\ x \mapsto x' := (x \perp B) \cap B$ is the orthogonal projection from x to B, and $\pi_B: B \to A;\ x' \mapsto x := (x' \perp A) \cap A$ is the orthogonal projection from x' to A. The composition map $\pi_{AB} = \pi_B \circ \pi_A: A \to A; x \mapsto x''$ of these projections is called an orthogonal zigzag operation (see Figure 3).

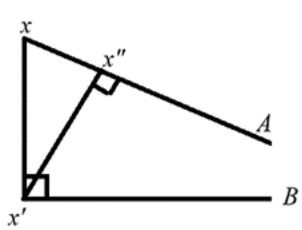

Figure 3: Zigzag Operation π_{AB}

In section 3, we investigated properties of the zigzag operation, and then in the subsequent sections, we apply it to characterize the relationship between lines. Also, in the case of common perpendicular lines, we present a novel classification of absolute planes. We summarize our results here (see theorems 3.1, 4.1, 4.3, 5.2, and corollary 5.3):

1. $A \cap B \neq \emptyset$ and $A \not\perp B$ if and only if π_{AB} has a unique fixed point $p := A \cap B$ and a contraction to p (see Figure 2, (1)).
2. $A \perp B$ if and only if π_{AB} is the constant map $\pi_{AB} = p := A \cap B$ (see Figure 2, (2)).
3. $A \cap B = \emptyset$ *and* $A^{\perp} = B^{\perp}$, i.e., A and B are coperpendicular if and only if $\pi_{AB} = id$ if and only if the absolute plane is singular (the congruence is Euclidean) (see Figure 2, (3)).
4. $A \cap B = \emptyset$ *and* $|A^{\perp} \cap B^{\perp}| = 1$, i.e., A and B exactly have one common perpendicular S if and only if π_{AB} has a unique fixed point $s := A \cap S$, and then the zigzag operation is a contraction to s if the congruence $\equiv$ is hyperbolic, and an extension from s if the congruence $\equiv$ is elliptic (see Figure 2, (4)).
5. $A \ddagger B$ (A and B are quasi-parallel) if and only if π_{AB} is a fixed-point-free isotone permutation (see Figure 2, (5)).

2 Notions and Known Results

In this section, we recall some main concepts and known results in absolute planes that we need in the following sections. Most are from [2].

For $(\boldsymbol{G} \times \boldsymbol{E} \times \boldsymbol{E})' := \{(L, a, b) \in \boldsymbol{G} \times \boldsymbol{E} \times \boldsymbol{E} : a, b \in \boldsymbol{E} \setminus L\}$, the function $\boldsymbol{\alpha}$ is defined as

$$\boldsymbol{\alpha}: (\boldsymbol{G} \times \boldsymbol{E} \times \boldsymbol{E})' \to \{-1,1\};\ (L, a, b) \mapsto (L|a, b)$$

is called an *order function* if the following conditions hold:

1. For any $L \in \boldsymbol{G}$ and $a, b \in \boldsymbol{E} \setminus L, (L|a, b) = (L|b, a)$.

2. For any $L \in \boldsymbol{G}$ and $a, b \in \boldsymbol{E} \setminus L$, $(L|a, b) \cdot (L|b, c) = (L|a, c)$.

Theorem 2.1 Suppose that $X, Y, Z, W \in \boldsymbol{G}$ are four distinct lines passing through a point p, and let a, b, c, and d be another points on X, Y, Z, and W other than p, respectively. Moreover, let $M, N \in \boldsymbol{G}$ with $M \neq N$ *and* $M^{\perp} \cap N^{\perp} \neq \emptyset$. Then:

1. $(X|b\,, c) \cdot (Y|c\,, a) \cdot (Z|a\,, b) = -1$ (cf., [2] (13.12.1)).
2. $(X|c\,, d) \cdot (Y|c\,, d) = (Z|a\,, b) \cdot (W|a\,, b)$ (cf., [2] (13.12.2)).
3. For any $m, p \in M$, we have $(N \,|\, m, p) = 1$ and $M \cap N = \emptyset$ *(cf.,*[2]*, 16.10).*
4. For any distinct points $x, y \in \boldsymbol{E}$ with $x, y \notin M$, if $m := \overline{x, y} \cap M \neq \emptyset$ then $(M|x, y) = (m|x, y)$ (cf., [2] (13.9)).

In the following, we recall and state some theorems on rectangular triangles and other particular configurations:

Let $\Delta(\alpha, \alpha', \alpha'')$ be a right-angled triangle in α'' with $A = \overline{\alpha'', \alpha}$, $X' = \overline{\alpha'', \alpha'}$ *and*, $X = \overline{\alpha', \alpha}$, and let $B := (\alpha' \perp X)$.

Theorem 2.2 *Let* $A \cap B = \emptyset$, *and for* $l \in A$ *let* $T := (l \perp X)$ *and* $t := T \cap X$, *moreover let* $H := (\alpha'' \perp X)$ *and* $h := H \cap X$. *Then:*

1. $T \cap B = \emptyset$ *and* $(\alpha'|t\,, \alpha)$=1.
2. $(B|\alpha, \alpha'')$=1.
3. $(h|\alpha, \alpha') = (H|\alpha, \alpha') = -1$.
4. *If* $l \in \overrightarrow{\alpha, \alpha''}$ *(hence* $(\alpha|l\,, \alpha'') = 1$*) then* $(\alpha|t\,, \alpha') = 1$ *and so* $(T|\alpha\,, \alpha') = (t|\alpha\,, \alpha')$=–1.
5. *If* $l \in \overrightarrow{\alpha, \alpha''}$, $c \in]\alpha'', \alpha'[$ *(hence* $(c|\alpha'', \alpha') = -1$*) and* $Z := \overline{c, \alpha}$ *then* $(Z|t\,, l) = -1$; *hence the lines T and Z intersect each other.*

Proof. 1. Since T and B are orthogonal to X, by part 3 of theorem 2.1, we have $T \cap B = \emptyset$. From $A \cap B = \emptyset$ and again part 3 of theorem 2.1 follows $(B|l, \alpha) = 1 = (B|t, l)$. Now by (**Z 2**), $(\alpha'|t\,, \alpha) = (B|t\,, \alpha) = (B|t, l) \cdot (B|l, \alpha) = 1 \cdot 1 = 1$.

2. Follows from part 3 of theorem 2.1, because $\alpha, \alpha'' \in A$ and $A \cap B = \emptyset$.

3. Follows from Theorem (20.5) of [2].

4. By part 3, $(h|\alpha\,, \alpha') = -1$ so $(\alpha|h\,, \alpha') = 1$ and $(\alpha|h\,, t) = (\alpha|\alpha'', l) = 1$. On the other hand, $(\alpha|\, t,\, \alpha')$ is equal to $(\alpha|h\,, \alpha') \cdot (\alpha|h\,, t) = 1 \cdot 1 = 1$. So together with part 1, we have $(t|\alpha\,, \alpha') = -1$.

5. We have $(Z|t\,, l) = (Z|l, \alpha'') \cdot (Z|\alpha'', \alpha') \cdot (Z|\alpha', t)$ which is equal to $(\alpha|\alpha'', l) \cdot (c|\alpha'', \alpha') \cdot (\alpha|\alpha', t) = 1 \cdot (-1) \cdot 1 = -1$.

From chapter 5 of [2], we recall the definition of a half-rotation (Halbdrehung):

For any two non-orthogonal lines A and B with $o \in A \cap B$, and $p \in \boldsymbol{E} \setminus \{o\}$, $P = \overline{o, p}$, $\widetilde{P^*} = \tilde{P} \circ \tilde{A} \circ \tilde{B}$, *and* $p^* = (p \perp P^*) \cap P^*$, the half-rotation η_{AB} in o is defined as follows (see Figure 4):

$$\eta_{AB}: E \to E; p \mapsto \begin{cases} o & p = o \\ p^* & p \neq o \end{cases}$$

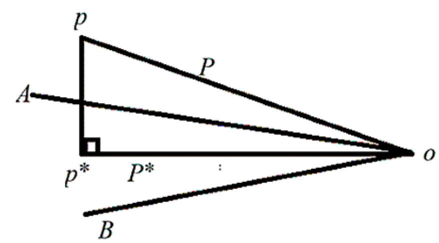

Figure 4: Half-rotation of η_{AB} about point o.

Proposition 2.3 A half-rotation η in point $o \in \boldsymbol{E}$ is a collinearity-preserving injection of $\boldsymbol{E}$ into $\boldsymbol{E}$ (therefore, if $X \in \mathbf{G}$ is a line, then η(X) is contained in exactly one line, which we denote by $\overline{\eta(X)}$). If $X, Y \in \mathbf{G}$ are two lines such that at least one of them contains the point o, then $X \perp Y$ if and only if $\overline{\eta(X)} \perp \overline{\eta(Y)}$ (cf., p. 134 of [2]).

A *motion* of an absolute plane is a permutation of the point set **E** that preserves the line structure **G**, the order **α**, and the congruence ≡ of **A**. The two main motions of an absolute plane are point reflections and line reflections.

For any $a \in \boldsymbol{E}$, the point reflection in a, denoted by $\tilde{a}$, is the map defined by

$$\tilde{a}: \boldsymbol{E} \to \boldsymbol{E};\ x \mapsto \tilde{a}(x) \text{ if } x \neq a \text{ and } x \mapsto a \text{ if } x = a$$

where a is the midpoint of x and $\tilde{a}(x)$.

The line reflection in $G \in \boldsymbol{G}$ is the map

$$\tilde{G}: \boldsymbol{E} \to \boldsymbol{E};\ x \mapsto \widetilde{x_G}(x) \text{ if } x \notin G \text{ and } x \mapsto x \text{ if } x \in G$$

where $x_G = \{x \perp G\} \cap G$. We set $\tilde{\boldsymbol{E}} := \{\tilde{p} : p \in \boldsymbol{E}\}$ *and* $\tilde{\boldsymbol{G}} := \{\tilde{L} : L \in \boldsymbol{G}\}$, the set $\boldsymbol{M}^{+} := \{\tilde{A} \circ \tilde{B} : A, B \in \boldsymbol{G}\} = \tilde{\boldsymbol{L}}^2$is the subgroup of proper motions (cf., 17.18 of [2]). We denote the set of involution motions by **J**. Clearly, $\tilde{\boldsymbol{E}}, \tilde{\boldsymbol{G}} \subset \boldsymbol{J}$.

Absolute planes are divided into two classes:
• **Singular absolute planes**, in which for all $a, b, c \in$**E**, $\tilde{a} \circ \tilde{b} \circ \tilde{c} \in \tilde{\boldsymbol{E}}$, or equivalently, rectangles exist, if and only if there are lines A and B such that $A^{\perp} = B^{\perp}$, if and only if the congruence is Euclidean [1,3].

• **Ordinary absolute planes**, in which there exist points $a, b, c \in$**E** such that $\tilde{a} \circ \tilde{b} \circ \tilde{c} \notin \tilde{\boldsymbol{E}}$, or equivalently, rectangles don't exist, if and only if for all lines A and B we have $A^{\perp} \neq B^{\perp}$, if and only if the congruence is hyperbolic or elliptic [1,3].

Now, we recall the notions of pencil and quasi-end. Let A and B be two distinct lines of an absolute plane; Then the set $\widetilde{A,B} := \{X \in \boldsymbol{G}: \tilde{A} \circ \tilde{B} \circ \tilde{X} \in \boldsymbol{J}\}$ is called a *pencil*. In the case when $A \ddagger B$ (A and B are quasi-parallel lines), we call it a *quasi-end* [5]. In general, there are three types of pencils [5]:

1. Point-pencil: $\hat{p} = \{X \in \boldsymbol{G}: p \in X\}$.
2. Orthogonal-Pencil: $A^{\perp} = \{X \in \boldsymbol{G}: X \perp A\}$.
3. Quasi-end: $\widetilde{A,B} := \{X \in \boldsymbol{G}: A \ddagger B \wedge \tilde{A} \circ \tilde{B} \circ \tilde{X} \in \boldsymbol{J}\}$.

Karzel and et al. [5] has classified general absolute planes in the most general case by the cardinality of quasi-ends that are incident with a line.

3 Orientation and the Orthogonal Zigzag Operation

In this section, we will define an orientation on a line. Then, we introduce the zigzag operation process and apply it to a general absolute plane by verifying its direction and fixed points. We will see that the zigzag operation plays a crucial role in characterizing geometric configurations determined by lines.

3. 1 Orientation

Starting from the order structure $\boldsymbol{\alpha}$ of an absolute plane [2], one can derive a betweenness relation on each line L and then induce the following *orientation* ♣ on the points of L (cf., [2] (13.3)):

For any pairs of $(m, n), (p, q)$ on L:

$(m,n)♣(p,q) \Leftrightarrow (p|m,q) = -(m|p,n)$ if $m \neq p$, and $(m|n,q) = 1$ if $m = p$.

An injection $\phi: L \rightarrow L$ is called an *isotone* if for any pair of points (m, n) on L we have $(m,n)♣\left(\phi(\mathrm{m}), \phi(\mathrm{n})\right)$, and an *antitone* if $(m,n)♣\left(\phi(n), \phi(m)\right)$.

We say that ϕ has a *direction* if ϕ is fixed-point-free and if $\forall m, n \in L: (m, \phi(m))$ ♣ $(n, \phi(n))$.

ϕ is called a *contraction to* p (an *extension from* p) if $\phi(p) = p$ and if for any $\lambda \in A \setminus \{p\}, \phi(\lambda) \in]p, \lambda[$ ($\lambda \in]p, \phi(\lambda)[$), respectively.

3. 2 Orthogonal Zigzag Operation

For $A, B \in \boldsymbol{G}$ and $x \in A$ we consider the maps

$\pi_A : A \rightarrow B;\ x \mapsto x' := (x \perp B) \cap B$(orthogonal projection from x to B)

$\pi_B : B \rightarrow A;\ x' \mapsto x'' := (x' \perp A) \cap A$(orthogonal projection from x' to A)

Then, the composition map $\pi_{AB} = \pi_B \circ \pi_A: A \rightarrow A; x \mapsto x''$ of these projections is called the orthogonal zigzag operation (see Figure 3).

Note that, if $A \perp B$ with $p = A \cap B$, then for any $x \in A, x' = x'' = p$. So, in this case, the zigzag operation is the constant map $\pi_{AB} = p$. Therefore, from now on, let $A, B \in \boldsymbol{G}$ with $A \neq B$ and $A \not\perp B$, and for each $\lambda \in A$, let X be the line perpendicular to B from $\lambda, \lambda' = \pi_A(\lambda) = X \cap B$ be the orthogonal projection from λ to B, $X' := (\lambda' \perp A)$, and $\lambda'' = X' \cap A = \pi_{AB}(\lambda)$.

The orthogonal zigzag operation has the following properties:

Theorem 3.1 *Let* λ *and* β *be two distinct points on line A. Then:*

1. $\lambda'' \neq \beta''$, *i.e.,* π_{AB} *is an injection, and* $(X|\beta,\beta') = (X'|\beta',\beta'') = 1$.
2. *If* $\lambda \notin B$, *then* $\left(B|\lambda,\lambda''\right) = 1$, *and if, moreover,* $\{s\}:=A\cap B\neq\emptyset$ *then* $\left(\lambda''|\lambda, s\right) = (X'|\lambda, s) = -1$.
3. *If* $\lambda \neq \lambda''$ *then* $(\lambda|\lambda'',\beta) = -(\lambda''|\lambda,\beta''), i.e., (\lambda,\beta)♣(\lambda'',\beta'')$.
4. *If* $\lambda = \lambda''$ *then* $(\lambda|\beta,\beta'') = 1, i.e., (\lambda,\beta)♣(\lambda'',\beta'') = (\lambda,\beta'')$.
5. π_{AB} *is an isotone injection of A into A and has a direction if for any distinct points* λ *and* β *on* A, $\lambda \neq \lambda''$ *and* $(\lambda|\lambda'',\beta) = -(\beta|\beta'',\lambda)$.
6. *If* $\{s\}:=A\cap B\neq\emptyset$, *then* π_{AB} *is a contraction to s and there are distinct points c and d on A with* $(c|c'',d) = (d|d'',c)$.
7. *If S is the coperpendicular of A and B, and* $\{o\} := S \cap A$, *then* π_{AB} *is a contraction to o if the congruence* $\equiv$ *is hyperbolic, and it is an extension from o if the congruence* $\equiv$ *is elliptic.*

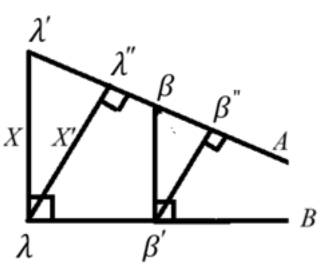

Figure 5: Notations used in Theorem 3.1.

Proof. 1. Since B is the coperpendicular of X and $\overline{\beta,\beta'}$, *by part 3 of Theorem 2.1, we have* $(X|\ \beta,\beta') = 1$. *Similarly, as A is the coperpendicular of X´ and* $\overline{\beta',\beta''}$, *we have* $(X'|\beta',\beta'')$= 1.

2. If $\lambda \neq \lambda''$, then $\Delta(\lambda,\lambda'',\lambda')$ is a right-angled triangle in λ'' and $B = (\lambda' \perp \overline{\lambda,\lambda''})$. Hence, by part 2 of Theorem 2.2 $(B|\ \lambda,\lambda'') = 1$. If $A \cap B \neq \emptyset$, then $\Delta(\lambda,\lambda',s)$ is a right-angled triangle in s and by part 3 of Theorem 2.2, $(\lambda''|\ \lambda, s) = (X'|\ \lambda, s) = -1$.

3. $\lambda'' \neq \lambda$ implies $\lambda'' \notin X$, $\lambda \notin X'$, and $\lambda'' \notin B$. Since λ and β are distinct, β' is not on $X \cup X'$. Therefore, we can form $(B|\ \lambda,\lambda'')$, $(X|\lambda'',\beta')$, $(X'|\ \lambda,\beta')$, and since $\beta' \in B$, $\lambda \in X$, $\lambda'' \in X'$, we have by Theorem 2.1,

$(B|\ \lambda,\lambda'') \cdot (X|\lambda'',\beta') \cdot (X'|\ \lambda,\beta') = -1$

Hence, by parts 1 and 2, $(X|\lambda'',\beta') = -(X'|\lambda,\beta')$ and

$$(\lambda|\lambda'',\beta) = (X|\lambda'',\beta) = (X|\lambda'',\beta') \cdot (X|\beta',\beta) = (X|\lambda'',\beta')$$

$= -(X'|\ \lambda,\beta') = -(X'|\ \lambda,\beta'') = -(\lambda''|\ \lambda,\beta'')$

Therefore $(\lambda\ ,\beta)♣(\lambda'',\beta'')$.

4. $\lambda = \lambda''$ implies either $\{s\} = A \cap B \neq \emptyset$ and so $\lambda = s$ or S is the coperpendicular of A and B and so $X = X' = S$. If $\lambda = s$, then $\lambda \in B$ and from $\beta \neq \lambda, \beta \notin B$ implying by part 2, $(\lambda\ |\beta'',\beta) = (B|\beta'',\beta) = 1$.

If X$= X' = S$, then by part 1,

$$\begin{aligned}(\lambda\ |\beta'',\beta) &= (X|\beta'',\beta) = (X|\beta'',\beta') \cdot (X|\beta',\beta) \\ &= (X'\ |\beta''\ ,\beta\ ) \cdot 1 = 1 \cdot 1 = 1.\end{aligned}$$

5. The first statement is a consequence of parts 1, 3, and 4. Also, from the definition of direction, it follows that π_{AB} has direction.

6. Firstly, we have $s'' = \pi_{AB}(s) = s$ and then by part 2, for any $\lambda \in A \setminus \{s\}$, $\pi_{AB}(\lambda) = \lambda'' \in]s,\lambda[$. Hence, π_{AB} is a contraction to s.

Now, let $c \in A \setminus \{s\}$ and $d := \tilde{s}(c)$. Then, $c \neq d$ and $(s|\ c, d) = -1$, and so $(c|\ s, d) = (d|s,c) = 1$.

From $c'' \in]s,c[$ we have $d'' \in]s,d[$, hence $(c''|c\ ,s) = (d''|d,s) = -1$.

By (**Z 3**), we have $(c|c'',s) = (d|d'',s) = 1$. Therefore,

$$(c|c'',d) = (c|c'',s) \cdot (c|s,d) = 1 \cdot 1 = 1$$

$$(d|d'',c) = (d|c,s) \cdot (d|s,d'') = 1 \cdot 1 = 1$$

Thus, we have $(c|c'',d) = (d|c,d'')$.

7. Clearly $o'' = \pi_{AB}(o) = o$. Now for any other point $\lambda \in A \setminus \{o\}, A = \overline{\lambda,o} \perp \overline{o,o'} \perp \overline{o',\lambda'} \perp \overline{\lambda',\lambda}, i.e., (\lambda',o',o,\lambda)$ is an LS-quadrangle. On the other hand, by definition of congruence for $\lambda'' = \pi_{AB}(\lambda) = \left(\lambda' \perp \overline{\lambda,o}\right) \cap \overline{\lambda,o}$, in the case of hyperbolic congruence $\lambda'' \in]o,\lambda[$, hence $(\pi_{AB}(\lambda)|o,\lambda) = -1$, i.e., by definition, it has a contraction to o. Similarly, we can get the result for the elliptic case.

Remark 3.1 *In part 7 of Theorem 3.1, we showed that if A and B are two distinct lines having a coperpendicular S and if* $s := S \cap A$, *then* π_{AB} *is a contraction to s if the congruence ≡ is hyperbolic and an extension from s if the congruence ≡ is elliptic. Now let* $\lambda,\beta \in A$ *with* $(s|\lambda,\beta) = -1$ *hence by* **(Z 3)**, $(\lambda|s,\beta) = (\beta|s,\lambda) = 1$ *then* $\lambda'' \in]\lambda,s[$ *hence* $(\lambda''|\lambda,s) = (\beta''|\beta,s) = -1$ *in the hyperbolic case implying* $(\lambda\ |\lambda'',s) = (\beta|\beta'',s) = 1$ *and* $(\lambda\ |\lambda'',s) = (\beta|\beta'',s) = -1$ *in the elliptic case. In the hyperbolic case,* $(\lambda|\lambda'',\beta) = (\lambda|\lambda'',s) \cdot (\lambda|s,\beta) = 1 \cdot 1 = 1 = (\beta|\beta'',\lambda)$ *and in the elliptic case,* $(\lambda|\lambda'',\beta) = (\lambda|\lambda'',s) \cdot (\lambda|s,\beta) = (-1) \cdot 1 = -1 = (\beta|\beta'',\lambda)$.

Now, we will consider the converse situation.

4 Criteria for Common Perpendicular

In this section, let ***A*** be an ordinary absolute plane. Hence, for any two distinct lines A and B, $A^{\perp} \neq B^{\perp}$.

Theorem 4.1 (Common perpendicular) Let A and B be two distinct lines with no intersection point, let $o \in A$ such that $O := (o\ \perp\ B) \pm A$, and let $Q \in B^{\perp} \setminus \{O\}$, $Q' := (o \perp Q)$, $q := Q' \cap Q$ and $A' \in$ **G** such that $\widetilde{A'} = \widetilde{A} \circ \widetilde{O} \circ \widetilde{Q'}$, $R := (q \perp A')$. Then $A^{\perp} \cap B^{\perp} \neq \emptyset$ if and only if $\{s\} := R \cap A \neq \emptyset$(i.e., $S := (s\ \perp\ A) \in A^{\perp} \cap B^{\perp}$) (see Figure 6).

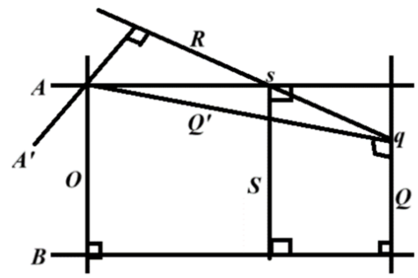

Figure 6: Notations used in Theorem 4.1.

Proof. Since A is ordinary, we have $|A^{\perp} \cap B^{\perp}| \leq 1$ and the lines O and Q are distinct and not perpendicular; hence they define a half-rotation $\eta := \eta_{OQ'}$ in o. Now, by proposition 2.3, from $Q = (q \perp Q')$ it follows $q \in \overline{\eta(Q)} = (\eta(q) \perp \overline{\eta(Q')})$. So, it turns out that $q = \eta(O) \cap \eta(Q)$, hence, as proved in [4]((vii), p. 96), the half-rotation η maps the pencil $B^{\perp}$ determined by the two lines O and Q onto the set of lines passing through the point q and the pencil $A^{\perp}$ onto the pencil $A'^{\perp}$. Now, if $A^{\perp} \cap B^{\perp}$ consists of a line S, then $\eta(S) = (q \perp A') = R$ and also there is a point $s = S \cap A$ such that $\eta(s) = (s \perp A') \cap A' \in R$, so $s = R \cap A$. This shows $A^{\perp} \cap B^{\perp} \neq \emptyset$.

Conversely, if the point $\{s\} := R \cap A$ exists, we can form the line $S ::= (s \perp A)$. Then, $S \perp B$ if and only if $\tilde{S} \circ \tilde{O} \circ \tilde{Q} \in \widetilde{\boldsymbol{G}}$ (cf., [2] (18.5)) if and only if $\widetilde{A'} \circ \widetilde{Q'} = \tilde{A} \circ \tilde{O}$. Thus, $S \in A^{\perp} \cap B^{\perp}$.

From now on, let A and B be two distinct lines with no intersection point, and for λ ∈ A, let λ′, X, X′, and λ′′ be defined according to the assumptions of Theorem 3.1. For distinct points λ and β on A, we show:

Theorem 4.2 *Let also* $\lambda \neq \lambda''$, $\beta \neq \beta''$ *and* $(\lambda|\lambda'',\beta) = (\beta|\beta'',\lambda) =: \varepsilon$. *Moreover, let* $Y := \overline{\beta,\beta'}, Y' := \overline{\beta',\beta''}, Z := (\lambda \perp Y), z := Z \cap Y, U := (\beta'' \perp Y), u = U \cap Y$, *and let* $L \in$ **G** *be the midline of the half-lines* $\overrightarrow{\lambda,z}$ *and* $\overrightarrow{\lambda,\lambda''}$, $z_l := \tilde{L}(z)$, *and* $T := (z_l \perp X)$. *Then:*

1. $(U|\beta,\beta') = (u|\beta,\beta') = -1$ *and so* $(\beta|u,\beta') = 1$.

2. $(B|z,\lambda)=1$.

3. $(\beta'|z,\beta)=(A|z,\beta)=(Y|\lambda,\beta'')=(y|\lambda,\beta'')=\varepsilon$ and $(Z|\beta',\beta'')=(Y'|z,\lambda)$.

4. $U\cap Z=B\cap Z=B\cap U=\emptyset$ and $(\beta|\lambda,\beta'')=(\beta|z,u)=\varepsilon$.

5. $(\beta|\beta',z)=(A|\beta',z)=(A|\lambda',z)=\varepsilon=-(Z|\beta,\beta')$.

6. $(Z|\lambda',\lambda'')=-1$, *i.e.* Z *is a line between the halflines* $\overrightarrow{\lambda,\lambda'}$ *and* $\overrightarrow{\lambda,\lambda''}$.

7. $Z\cap T\neq\emptyset$.

Proof.

1. Since $\Delta(\beta',\beta',\beta)$ is a right-angled triangle in β'', we have by 3 of Theorem 2.2, $(U|\beta,\beta')=-1$. Now, by (**Z 3**), $(\beta|u,\beta')=1$.

2. Since B, $Z\perp Y$ and $\lambda\notin B$, we have $B\cap Z=\emptyset$, hence by part 3 of Theorem 2.1, $(B|z,\lambda)=1$.

3. Suppose $(B|z,\beta)=(\beta'|z,\beta)=-1$. Then by (2), $(B|\lambda,\beta)=(B|\lambda,z)\cdot(B|z,\beta)=1\cdot(-1)=-1$ and so $B\cap]\lambda,\beta[\neq\emptyset$ hence $B\cap A\neq\emptyset$ a contradiction to $B\cap A=\emptyset$.

4. From $U,Z,B\perp Y$, and by $\beta\notin Z$ follows $U\cap Z=\emptyset$, by $x\notin B$, $B\cap Z=\emptyset$ and by $\beta''\notin B$, $B\cap U=\emptyset$. For $Y^{\circ}:=(\beta\perp Y)$, by Y°, Z, $U\perp Y$ we have $(Y^{\circ}|\lambda,z)=\left(Y^{\circ}|\beta'',u\right)=1$, hence

$$(\beta|\beta'',\lambda)=(Y^{\circ}|\beta'',\lambda)=(Y^{\circ}|\beta'',u)\cdot(Y^{\circ}|u,z)\cdot(Y^{\circ}|z,\lambda)=(Y^{\circ}|u,z)=(\beta|u,z).$$

5. By part 4, $(\beta|\lambda,\beta'')=(\beta|z,u)=\varepsilon$ and so together with part 1, $(A|z,\beta')=(\beta|z,\beta')=(\beta|z,u)\cdot(\beta|u,y')=\varepsilon\cdot1=\varepsilon$. By $A\cap B$, $(A|\lambda',\beta')=1$.

If $(A|z,\beta')=(\beta|z,\beta')=\varepsilon=-1$ then by (**Z 3**), $(z|\beta,\beta')=(Z|\beta,\beta')=1=-\varepsilon$. Now by part 3, $(\beta'|z,\beta)=1$ so if $(A|z,\beta')=(\beta|z,\beta')=\varepsilon=1$ then by (**Z 3**), $(z|\beta,\beta')=(Z|\beta,\beta')=-1=-\varepsilon$. Thus $\varepsilon=-(Z|\beta,\beta')$.

6. Since $Z,B\perp Y$ then $Z\cap B=\emptyset$ and so $(Z|\beta',\lambda')=1$. Also, $(Z|\beta,\lambda'')=$

$(\lambda|\beta,\lambda'')=\varepsilon$ and by (1), $(z|\beta,\beta')=-\varepsilon$. Therefore, $(Z|\lambda',\lambda'')=(Z|\lambda',\beta')\cdot(Z|\beta',\beta)\cdot(Z|\beta,\lambda'')=1(-\varepsilon)\,\varepsilon=-1$.

7. Since L is the midline of the halflines $\overrightarrow{\lambda,\lambda''}$ and $\overrightarrow{\lambda,z}$, we have $(L|\lambda'',z)=-1$, and from $z_l=\tilde{L}(z)$ *we can have* $(L|z,z_l)=-1$. So, $(\lambda|\lambda'',z_l)=(L|\lambda,z_l)$, *and we have*:

$(L|\lambda,z_l)=(L|\lambda'',z)\cdot(L|z,z_l)=(-1)\cdot(-1)=1.$

Therefore, $z_l\in\overrightarrow{\lambda,\lambda''}$. Now by part 6, $(Z|\lambda',\lambda'')=-1$, thus by part 5 of Theorem 2.2 we conclude $Z\cap T\neq\emptyset$.

Theorem 4.3 Let A and B be two distinct lines with no intersection point. Suppose that $x,y\in A$, $x'=(x\perp B)\cap B$, $y'=(y\perp B)\cap B$, $x''=(x'\perp A)\cap A$, $y''=(y'\perp A)\cap A$, $X=(x\perp B)$, $X'=(x'\perp A)$, $Y=(y\perp B)$, $Y'=(y'\perp A)$, and also, for $Z=(x\perp Y)$ *and* $z=Z\cap Y$, let $L\in\boldsymbol{G}$ be the midline of the halflines $\overrightarrow{x,z}$ and $\overrightarrow{x,x''}$ (see Figure 7). Then:

1. *There is a line* $A'\in\mathbf{G}$ *such that* $\widetilde{A'}=\tilde{A}\circ\tilde{X}\circ\tilde{Z}$ *and we can form* $V::=(z\perp A')$ *and* $z':=A'\cap V$.

2. $A'=\tilde{L}(X)$.

3. $V\cap A\neq\emptyset$, *and so, if* $\{w\}::=V\cap A$ *then* $\{(w\perp A)\}=A^{\perp}\cap B^{\perp}$ *is the common perpendicular of the lines A and B*

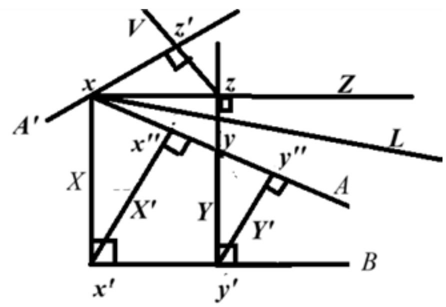

Figure 7: Notations used in Theorem 4.3.

Proof. 1. Since $x\in A,X,Z$, the statement follows from (17.13) of [2].

2. By definition of L, $\tilde{L}\,(\overrightarrow{x,z})=\overrightarrow{x,x''}$ and from $x,z\in Z$ and $x,x''\in A$ we have $\tilde{L}(Z)=A$, implying $\tilde{A}=\tilde{L}\circ\tilde{Z}\circ\tilde{L}$. Therefore, $\widetilde{A'}=\tilde{A}\circ\tilde{X}\circ\tilde{Z}=\left(\tilde{L}\circ\tilde{Z}\circ\tilde{L}\right)\circ\tilde{X}\circ\tilde{Z}=\tilde{L}\circ\tilde{X}\circ\tilde{L}=\widetilde{\tilde{L}(X)}$, hence $A'=\tilde{L}(X)$.

3. By part 7 of Theorem 4.2, let $\{c\}:=Z\cap T\neq\emptyset$, hence $T=\overline{z_l,c}$ and $w:=\tilde{L}(c)$. So, $w\in A$ and $\tilde{L}(T)=\tilde{L}(\overline{z_l,c})=\overline{\tilde{L}(z_l),\tilde{L}(c)}=\overline{z,w}$. On the other hand, since $T\perp X$ then $\tilde{L}(T\perp X)=\tilde{L}(T)\perp\tilde{L}(X)=\overline{z,w}\perp A'$. Hence, $\tilde{L}(T)=(z\perp A')=\overline{z,x}=V$, and so $\{w\}=V\cap A$. Therefore, by Theorem 4.1, $\{(w\perp A)\}=A^{\perp}\cap B^{\perp}$ is the common perpendicular of the lines A and B.

5 Criteria for Quasi-Parallelism

In this section, let $A,B\in\boldsymbol{G}$. We apply the zigzag operation properties to characterize quasi-parallel lines in a general absolute plane. Firstly, we prove:

Theorem 5.1 *Let A and B are two distinct lines. Then*:

$A\ddagger B$ *if and only if* $A\pm B\wedge\forall x,y\in A, x\neq y:\left(x|x'',y\right)=-\left(y|y'',x\right)$.

Proof. Let $A\ddagger B$.

By definition of quasi-parallel lines, $A\cap B=\emptyset$, so $A\pm B$. Now if $(x|x'',y)=(y|y'',x)$ then by part 4 of Theorem 4.3, $A^{\perp}\cap B^{\perp}\neq\emptyset$, which is a contradiction to $A\ddagger B$.

Conversely, if $A \cap B \neq \emptyset$ then by 6 of Theorem 3.1 there are distinct points x and y on A such that $(x|y, x'') = (y|x, y'')$ which is a contradiction, so $A \cap B = \emptyset$.

If $A^{\perp} \cap B^{\perp} \neq \emptyset$, then by Remark 3.1 there are distinct points x and y on A with $(x|y, x'') = (y|x, y'')$ which is a contradiction, hence A and B have no coperpendicular. Thus $A \ddagger B$.

Theorem 5.2 *If $A \pm B$, then the orthogonal zigzag operation π_{AB} is an isotone injection of A into A, and we have:*

1. $\pi_{AB} = id_A \Leftrightarrow A^{\perp} = B^{\perp}$.
2. $x \in Fix(\pi_{AB}) \Leftrightarrow x \in A \cap B \vee (x \perp A) = (x \perp B)$.
3. *If $A \neq B$, $A \pm B$, and $\{o\} = A \cap B \neq \emptyset$, then π_{AB} is a contraction to o, and if η_{AB} denotes the half-rotation about o [2] determined by A and B then $\pi_{AB} = \eta_{BA} \circ \eta_{AB}|_A$.*
4. *If $A \neq B, A^{\perp} \cap B^{\perp} \neq \emptyset, A^{\perp} \neq B^{\perp}$, $\{C\} = A^{\perp} \cap B^{\perp}$, and $c := C \cap A$ then π_{AB} is a contraction to c in the hyperbolic case and an extension from c in the elliptic case.*
5. *If $A \ddagger B$, then π_{AB} is fixed-point-free.*

Proof. From part 5 of Theorem 3.1, π_{AB} is an isotone injection.

Parts 1 and 2 are a consequence of the definition of the zigzag operation.

3. From $A \cap B \neq \emptyset$ follows that $A^{\perp} \cap B^{\perp} = \emptyset$, and so by part 2, o is the only fixed point of π_{AB}. For any $x \in A$ with $x \neq o$, the triple $(x, \pi_B(x), o)$ forms a right-angle triangle in $\pi_B(x)$. Therefore, by (20.5) of [2], $\pi_{AB}(x) \in]o, x[$, i.e., π_{AB} is a contraction to o.

4. From $\{C\} = A^{\perp} \cap B^{\perp}$ and 2., $c := C \cap A$ is the only fixed point of π_{AB}. If $d := C \cap B$ then for each $x \in A \setminus \{c\}, (x, c, d, \pi_B(x)) \in LS$. Therefore, if the congruence is hyperbolic, we have $\pi_{AB}(x) \in]c, x[$, i.e., π_{AB} is a contraction to c, and if the congruence is elliptic, then $x \in]c, \pi_{AB}(x)[$, hence π_{AB} is an extension from c.

5. From part 2 it follows that π_{AB} is fixed-point-free.

From Theorem 5.2 follows:

Corollary 5.3 In any general absolute plane, any two distinct lines A and B are quasi-parallel if and only if the zigzag operation π_{AB} is a fixed-point-free isotone permutation.

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