Available online at www.garmian.edu.krd

Passer Journal of Basic and Applied Sciences

Passer 7 (Issue 2) (2025) 1033-1042

Original Article

Study of Homotopy Perturbation Transform for Solving Cubic-Quintic Nonlinear Schrödinger Equation

Kazheen Hasan Omar * [1], Fadhil Hamead Easif [1]

[1] *Department of Mathematics, College of Science, University of Zakho, Zakho 42002, Kurdistan Region, Iraq.*

Received 14 April 2025; revised 31 August 2025;
accepted 10 September 2025; available online 24 September 2025

DOI: 10.24271/PSR.2025. 516820.2075

ABSTRACT

This paper is devoted to investigating and comparing the homotopy perturbation method and the homotopy perturbation transform method for solving the cubic quintic nonlinear Schrödinger equation. Both proposed methods are iterative schemes to find solutions without discretization, linearization, or restrictive assumptions. The solutions obtained in this research are approximate analytical series solutions to the cubic-quintic nonlinear Schrödinger equation. Using the homotopy perturbation method and the homotopy perturbation transform method. These methods generate rapidly converging series that approximate the behavior of the wave function over time. In addition, the current results have been shown graphically and in a table. The results demonstrate that the homotopy perturbation transform method provides more accurate and efficient solutions than the standard homotopy perturbation method.

Keywords: Cubic-quintic nonlinear Schrödinger equation, Homotopy perturbation method, Homotopy perturbation transform method, Numerical solution.

1 Introduction

Numerous scientific and technical fields often use the Schrödinger equation [1,2]. Numerous theoretical and empirical investigations of models derived from the nonlinear Schrödinger equation (NLSE) have occurred in recent years. The cubic-quintic nonlinear Schrödinger (CQNLS) model is one of the most straightforward expansions of the cubic NLS. The cubic-quintic nonlinear Schrödinger equation (CQNLSE) is a general mathematical model that describes a wide range of physical phenomena and approximate more complicated systems, such as Bose-Einstein condensates (BEC) [3], nuclear physics [4], condensed matter physics [5], plasma physics [6], nonlinear optics [7], quantum mechanics [8,9], fluid mechanics [10], etc [11]. Researchers have employed various solution approaches to study the CQNLSE extensively. For example, by assuming a trial solution, Serkin discovered the novel stable bright and dark soliton management regimes for the CQNLSE model [12]. He found the explicit spatial self-similar, dazzling, and dark soliton solutions of the CQNLSE using distributed coefficients and an external potential [13]. Caplan examined the existence, interactions, and stability of solitary vortices in the two-dimensional CQNLSE using analytical and numerical techniques [14]. Hafez et al. used the $\exp(-\varphi(\xi))$ approach in their work to extract both singular and periodic solutions [15].

* Corresponding author
E-mail address kazheen.omar@uoz.edu.krd (Instructor).
Peer-reviewed under the responsibility of the University of Garmian.

A Chinese mathematician named Ji-Huan He [16] initially proposed the HPM in 1998. It has been further developed and is now being used by scientists and engineers to address various technical and scientific challenges [17]. This method can provide analytical, approximate, and exact solutions to nonlinear equations simply and elegantly, without the need to transform the equation or linearize the problem. In addition to avoiding physically impossible assumptions, it guarantees great accuracy with minimum computation. Many researchers have contributed, such as Beléndez et al. [18], Ariel et al. [19], Cveticanin [20], Öziş and Yildirim [21], to name a few, and it has developed into a thoroughly developed theory. Large classes of nonlinear problems can be solved successfully, efficiently, and accurately using this approach; in many cases, just one or two iterations yield highly precise results [22].

The HPM is a powerful technique that combines the classical perturbation method with homotopy concepts to solve linear and nonlinear equations without relying on small parameters, linearization, or discretization [23]. This approach provides accurate analytical approximate solutions to diverse problems in the applied sciences, including the Kuramoto-Sivashinsky equation [24], fractional wave equation [25], Fredholm integral equations [26], and Fisher equation [27]. Its flexibility and simplicity make it broadly applicable in various scientific and engineering fields.

Using He's polynomials simplifies the handling of nonlinear terms. Many researchers have combined the HPTM in conjunction with the Laplace transform. Swielam [28] obtained precise solutions for several coupled nonlinear differential equations. Singh applied the HPTM to solve both the linear and nonlinear Klein–Gordon equations [29]. Elbadri does a good job of comparing HPM with HPTM [17].

This paper consists of the following sections: Section 2 presents the CQNLSE mathematical model. Section 3 describes the basic ideas of the HPM and HPTM. Section 4 illustrates the derivation of both methods for the CQNLSE, while Section 5 provides a numerical example demonstrating their efficiency and accuracy.

2 Mathematical Model

2. 1 Cubic-Quintic Nonlinear Schrödinger Equation

Consider the CQNLSE [30].

$$iU_t = \frac{\beta_1}{2} U_{XX} + i\frac{\beta_2}{6} U_{XXX} + \frac{\beta_3}{24} U_{XXXX} - \alpha_1 |U|^2\, U - \alpha_2 |U|^4\, U. \quad (1)$$

Where $i = \sqrt{-1},\ \beta_1, \beta_2, \beta_3, \alpha_1$ and $\alpha_2 \in \mathcal{R}$.

Where $U(X,t)$ is the slowly changing electric field envelope, X is the distance along the velocity dispersion direction, t is the time, β_1 is the second-order dispersion is associated with the group velocity dispersion, β_2 is the third-order dispersion, β_3 is the fourth-order dispersion, and α_1 and α_2 are the coefficients of the cubic-quintic terms.

2. 2 Bright Soliton

The bright soliton for the CQNLSE (1) is [30]:

$$U(X,t) = sech(\frac{4\,t}{3} - X) e^{i(-\frac{t}{6} - X)} \quad (2)$$

A bright soliton is an analytical solution of nonlinear waves, such as the Schrödinger equation, with focusing nonlinear wave equations. It is a localized wave that balances nonlinear self-focusing and dispersion to keep its form. Bright solitons, which are frequently described using the hyperbolic secant (sech) functions, decrease to zero at infinity. These exact forms, which are acquired via techniques such as ansatz or the inverse scattering transform, are regarded as analytical rather than numerical.

3 Description of the method

In this section, we will show the main ideas of the HPM and HPTM.

3. 1 Basic idea of Homotopy Perturbation Method

To illustrate the basic idea of this method, we consider the following equation [31]:

$$A(U) - F(\mathfrak{r}) = 0, \quad \mathfrak{r} \in \eta. \quad (3)$$

With the boundary condition of:

$$B\left(U, \frac{\partial U}{\partial n}\right) = 0, \qquad \mathfrak{r} \in \varpi. \quad (4)$$

Where A is a general differential operator, and B a boundary operator, $F(\mathfrak{r})$ as a known analytical function, and ϖ is the boundary of the domain η.

It can be divided into two parts, $\mathbb{L}$ and $\mathbb{N}$, where $\mathbb{L}$ is a linear, and $\mathbb{N}$ a nonlinear operator. Equation (3) can be, therefore, written as follows:

$$\mathbb{L}(U) + \mathbb{N}(U) - F(\mathfrak{r}) = 0, \quad \mathfrak{r} \in \eta, \quad (5)$$

Homotopy perturbation structure is shown as follows:

$$H(v,\rho) = (1-\rho)[\mathbb{L}(v) - \mathbb{L}(U_0)] + \rho[A(v) - F(\mathfrak{r})] = 0, \quad \rho \in [0,1], \quad \mathfrak{r} \in \eta, \quad (6)$$

Or

$$H(v,\rho) = \mathbb{L}(v) - \mathbb{L}(U_0) + \rho\, \mathbb{L}(U_0) + \rho[\mathbb{N}(v) - F(\mathfrak{r})] = 0, \quad \rho \in [0,1], \ \mathfrak{r} \in \eta. \quad (7)$$

Where

$$v(\mathfrak{r},\rho): \eta \times [0,1] \to \mathcal{R}. \quad (8)$$

In equations (6) and (7), ρ is an embedding parameter that belongs to the interval $[0,1]$, and U_0 is the first approximation that satisfies the boundary condition. We assume that the solutions to equations (6) and (7) can be expressed as a power series in ρ, as follows:

$$v = v_0 + \rho v_1 + \rho^2 v_2 + \cdots \quad (9)$$

Setting $\rho = 1$ yields the approximate solution to equation (9)

$$U = \lim_{\rho \to 1} v = v_0 + v_1 + v_2 + \cdots \quad (10)$$

Which is convergent [32].

3. 2 Basic idea of Homotopy Perturbation Transform Method

To illustrate the basic idea of the HPTM, we examine the following nonlinear differential equation with initial conditions of the form [33].

$$\mathfrak{D}U(X,t) + \mathbb{L}U(X,t) + \mathbb{N}U(X,t) = 0. \quad (11)$$

The initial condition (I.C.) is applied

$$U(X,0) = h(X). \quad (12)$$

Where $\mathfrak{D}$ is the first linear differential operator, concerning t. The linear and nonlinear differential operators about X are denoted by $\mathbb{L}$ and $\mathbb{N}$, respectively. When we apply the Laplace transform (denoted by $\mathfrak{L}$ in this work) to both sides of equation (11), we obtain:

$$\mathfrak{L}[\mathfrak{D}U(X,t)] + \mathfrak{L}[\mathbb{L}U(X,t)] + \mathfrak{L}[\mathbb{N}U(X,t)] = 0. \quad (13)$$

By using the Laplace transform differentiation property, we obtain

$$\mathfrak{L}[U(X,t)] = \frac{h(X)}{s} - \frac{1}{s}\mathfrak{L}[\mathbb{L}U(X,t)]\frac{1}{s}\mathfrak{L}[\mathbb{N}U(X,t)]. \quad (14)$$

When the Laplace inverse is applied to both sides of equation (14), it yields

$$U(X,t) = H(X,t) - \mathfrak{L}^{-1}\left[\frac{1}{s}\mathfrak{L}[\mathbb{L}U(X,t) + \mathbb{N}U(X,t)]\right]. \quad (15)$$

In this case, $H(X,t)$ stands for the term that results from the source term and the specified beginning circumstances, and using the HPM as:

$$U(X,t) = \sum_{K=0}^{\infty} \rho^K U_K(X,t). \quad (16)$$

Thus, it is possible to break down the nonlinear phrase as

$$\mathbb{N}U(X,t) = \sum_{K=0}^{\infty} \rho^K H_K(U). \quad (17)$$

Regarding a few He's polynomials $H_K(U)$ provided by

$$H_K(U_0,U_1,\dots,U_K) = \frac{1}{K!}\frac{\partial^K}{\partial\rho^K}[\mathbb{N}(\sum_{m=0}^{\infty} \rho^m U_m)]_{\rho=0}, \quad K = 0,1,2,3. \quad (18)$$

When substituting equations (16) and (17) in equation (15), we obtain

$$\sum_{K=0}^{\infty} \rho^K U_K(X,t) = H(X,t) - \rho\left(\mathfrak{L}^{-1}\left[\frac{1}{s}\mathfrak{L}[\mathbb{L}\sum_{K=0}^{\infty} \rho^K U_K(X,t) + \sum_{K=0}^{\infty} \rho^K H_K(U)]\right]\right). \quad (19)$$

This method uses He's polynomials to combine the HPM with the Laplace transform. The following estimates can be derived by comparing the coefficients of like powers of ρ.

$$\rho^0 : U_0(X,t) = H(X,t), \quad (20)$$

$$\rho^1 : U_1(X,t) = -\mathfrak{L}^{-1}\left[\frac{1}{s}\mathfrak{L}[\mathbb{L}U_0(X,t) + H_0(U)]\right], \quad (21)$$

$$\rho^2 : U_2(X,t) = -\mathfrak{L}^{-1}\left[\frac{1}{s}\mathfrak{L}[\mathbb{L}U_1(X,t) + H_1(U)]\right], \quad (22)$$

$\rho \to 1$, then the form of the approximate solution of equation (16) becomes.

$$U(X,t) = U_0(X,t) + U_1(X,t) + U_2(X,t) + \cdots \quad (23)$$

The above series solutions generally converge very rapidly [34].

4 Illustrative Application

In this section, the above methods will be applied to the CQNLSE.

4. 1 Derivation of HPM for Cubic-Quintic Nonlinear Schrödinger Equation

To solve equation (1) using HPM, we first create the following homotopy:

$$H(v,\rho) = (1-\rho)\left(i\frac{\partial v}{\partial t}\right) + \rho\left[i\frac{\partial v}{\partial t} - \frac{\beta_1}{2}\frac{\partial^2 v}{\partial X^2} - i\frac{\beta_2}{6}\frac{\partial^3 v}{\partial X^3} - \frac{\beta_3}{24}\frac{\partial^4 v}{\partial X^4} + \alpha_1 \bar{v}\, v^2 + \alpha_2 \bar{v}^2\, v^3\right] = 0. \quad (24)$$

Where

$|U|^2\, U = U^2\, \bar{U}$ and $|U|^4\, U = U^3\, \bar{U}^2$,

$\bar{U}$ is the conjugate of U [31].

That is

$$H(v,\rho) = i\frac{\partial v}{\partial t} - \frac{\beta_1}{2}\rho\frac{\partial^2 v}{\partial X^2} - i\frac{\beta_2}{6}\rho\frac{\partial^3 v}{\partial X^3} - \frac{\beta_3}{24}\rho\frac{\partial^4 v}{\partial X^4} + \alpha_1\rho\bar{v}\, v^2 + \alpha_2\, \rho\, \bar{v}^2\, v^3 = 0. \quad (25)$$

Using the same powers of ρ to replace v in equation (9) and equating the terms in equation (25), we get

$$\rho^0 : \ i\frac{\partial v_0}{\partial_t} = 0, \quad v_0(X,0) = h(X), \quad (26)$$

$$\rho^1 : \ i\frac{\partial v_1}{\partial_t} - \frac{\beta_1}{2}\frac{\partial^2 v_0}{\partial X^2} - i\frac{\beta_2}{6}\frac{\partial^3 v_0}{\partial X^3} - \frac{\beta_3}{24}\frac{\partial^4 v_0}{\partial X^4} + \alpha_1\bar{v}_0\, v_0^2 + \alpha_2\, \bar{v}_0{}^2\, v_0^3 = 0, \quad v_1(X,0) = 0, \quad (27)$$

$$\rho^2 : i\frac{\partial v_2}{\partial_t} - \frac{\beta_1}{2}\frac{\partial^2 v_1}{\partial X^2} - i\frac{\beta_2}{6}\frac{\partial^3 v_1}{\partial X^3} - \frac{\beta_3}{24}\frac{\partial^4 v_1}{\partial X^4} + \alpha_1\bar{v}_1 v_0^2 + 2\alpha_1 v_0\bar{v}_0 v_1 + 3\alpha_2\bar{v}_0^2 v_0^2 v_1 + 2\alpha_2 v_0^3\bar{v}_0\bar{v}_1 = 0, \ v_2(X,0) = 0. \quad (28)$$

The results of solving these equations are v_0, v_1, v_2, and so on. Consequently, we obtain

$$U = \sum_{m=0}^{k} v_m = v_0 + v_1 + v_2 + \cdots + v_k \quad (29)$$

The series (29) is convergent for most cases; some criteria are suggested for the convergence of the series (29), in our equation, in [23].

4.2 Derivation of HPTM for Cubic-Quintic Nonlinear Schrödinger Equation

Consider the CQNLSE (1) with the initial condition. When the Laplace transform is performed on both sides of equation (1), the result is

$$\mathfrak{L}[iU_t] = \mathfrak{L}\left[\frac{\beta_1}{2}U_{XX} + i\frac{\beta_2}{6}U_{XXX} + \frac{\beta_3}{24}U_{XXXX} - \alpha_1|U|^2\, U - \alpha_2|U|^4\, U\right]. \quad (30)$$

By using the Laplace transform differentiation feature, we have

$$\mathfrak{L}[U(X,t)] = \frac{1}{s}U(X,0) + \frac{1}{s}i\mathfrak{L}\left[-\frac{\beta_1}{2}U_{XX} - i\frac{\beta_2}{6}U_{XXX} - \frac{\beta_3}{24}U_{XXXX} + \alpha_1|U|^2 U + \alpha_2|U|^4\, U\right]. \quad (31)$$

The implication of an inverse Laplace transform would be

$$U(X,t) = U(X,0) + \mathfrak{L}^{-1}\left[\frac{1}{s}i\mathfrak{L}\left[-\frac{\beta_1}{2}U_{XX} - i\frac{\beta_2}{6}U_{XXX} - \frac{\beta_3}{24}U_{XXXX} + \alpha_1 U^2\bar{U} + \alpha_2\, U^3\bar{U}^2\right]\right]. \quad (32)$$

This method is described by mixing the HPM via He's polynomials with the Laplace transform. The following estimations might be obtained by comparing the coefficients of like powers of ρ.

$$\rho^0: U_0(X,t) = U(X,0), \tag{33}$$

$$\rho^1: U_1(X,t) = \mathfrak{L}^{-1}\left[-\frac{\beta_1}{2s}\, i\mathfrak{L}(U_0)_{XX} + \frac{\beta_2}{6s}\, \mathfrak{L}(U_0)_{XXX} - \frac{\beta_3}{24s}\, i\mathfrak{L}(U_0)_{XXXX} + \frac{\alpha_1}{s}\, i\mathfrak{L}\big(H_0(U)\big) + \frac{\alpha_2}{s}\, i\mathfrak{L}\big(G_0(U)\big)\right], \tag{34}$$

$$\rho^2: U_2(X,t) = \mathfrak{L}^{-1}\left[-\frac{\beta_1}{2s} i\mathfrak{L}(U_1)_{XX} + \frac{\beta_2}{6s}\mathfrak{L}(U_1)_{XXX} - \frac{\beta_3}{24s}\, i\mathfrak{L}(U_1)_{XXXX} + \frac{\alpha_1}{s} i\mathfrak{L}\big(H_1(U)\big) + \frac{\alpha_2}{s}\, i\mathfrak{L}\big(G_1(U)\big)\right]. \tag{35}$$

$$U = \sum_{m=0}^{K} U_m = U_0 + U_1 + U_2 + \cdots + U_K \tag{36}$$

The results will be U_0, U_1, U_2 once solving these equations, to get

5 Numerical Results

In this section, we use the methods from the previous part to solve the CQNLS problem numerically. We then compare the results with the exact solutions.

5. 1 Solving Cubic-Quintic Nonlinear Schrödinger Equation with the Use of HPM

Consider the CQNLSE (1) if we take the arbitrary constants $\beta_1 = \beta_2 = \beta_3 = \alpha_1 = \alpha_2 = 1$, [30].

With the I.C. of

$$U(X,0) = sech(X)\, e^{-iX} \text{ [24]}. \tag{37}$$

The exact solution of a problem is given by equation (2), and equations (26-28), we obtain

$$v_0(X,t) = sech(X)\, e^{-iX}, \tag{38}$$

$$v_1(X,t) = t\Big(i\frac{7}{6} e^{-Xi}\, sech(X) + i\, e^{-Xi}\, sech^3(X) + i\, e^{-Xi}\, sech^5(X) - i\frac{1}{3} e^{-Xi}\, tanh^2(X)\, sech(X) - i\, e^{-Xi}\, tanh^4(X)\, sech(X) + \frac{4}{3}\, e^{-Xi}\, tanh(X)\, sech(X)\Big), \tag{39}$$

$$v_2(X,t) = \frac{t^2 e^{-Xi} e^{Xi}}{36\left(e^{2X}+1\right)^7}\big(16 + e^{4X}(-4528+33999\,i) + e^{8X}(4528+33999\,i) + e^{10X}(3776-4506\,i) + e^{12X}(-16-63\,i) + e^{2X}(-3776-4506\,i) - 63\,i\big)i. \tag{40}$$

Next, using the second-order approximation, the approximation solution is $U(X,t) = v_0 + v_1 + v_2$.

$$U(X,t) = e^X\left(\frac{\begin{array}{c} 432\,e^{2X} + 1080\,e^{4X} \\ +1440\,e^{6X} + 1080\,e^{8X} \\ +432\,e^{10X} + 72\,e^{12X} \\ -84\,t^2 e^{6X} + 72 \\ +t(-96-12\,i) \\ +t\,e^{2X}(-384+888\,i) \\ +t\,e^{4X}(-480+3660\,i) \\ +5520\,t\,e^{6X}\,i \\ +t\,e^{8X}(480+3660\,i) \\ +t\,e^{10X}(384+888\,i) \\ +t\,e^{12X}(96-12\,i) \\ +t^2 e^{2X}(4506-3776\,i) \\ +t^2 e^{4X}(-33999-4528\,i) \\ +t^2 e^{8X}(-33999+4528\,i) \\ +t^2 e^{10X}(4506+3776\,i) \\ +t^2 e^{12X}(63-16\,i) \\ +t^2(63+16\,i) \end{array}}{\begin{array}{c} 36\,e^{Xi} + \\ 252\,e^{X(2+i)} \\ +756\,e^{X(4+i)} \\ +1260\,e^{X(6+i)} \\ +1260\,e^{X(8+i)} \\ +756\,e^{X(10+i)} \\ +252\,e^{X(12+i)} \\ +36\,e^{X(14+i)} \end{array}}\right). \tag{41}$$

5. 2 Solving Cubic-Quintic Nonlinear Schrödinger Equation with the Use of HPTM

Consider the CQNLSE (1) with the I.C. equation (37), and by solving the HPTM, we get equations (34-36) when solved, which give:

$$U_0(X,t) = sech(X)\, e^{-iX}, \tag{42}$$

$$U_1(X,t) = \frac{t\, e^{X(1-i)}\left(78\, e^{2X} i + e^{4X}(8-i) - 8 - i\right)}{3\left(e^{2X}+1\right)^3}, \tag{43}$$

$$U_2(X,t) = \frac{t^2 e^{-Xi}}{72}\big(63\, sech(X) - 16i\tanh(X)\, sech(X) + 1032\, sech^3(X) + 960\, i\tanh(X)\, sech^3(X) - 3240\, sech^5(X) + 1680\, sech^7(X)\big), \tag{44}$$

These equations, when solved, provide the values of U_0, U_1, and U_2.

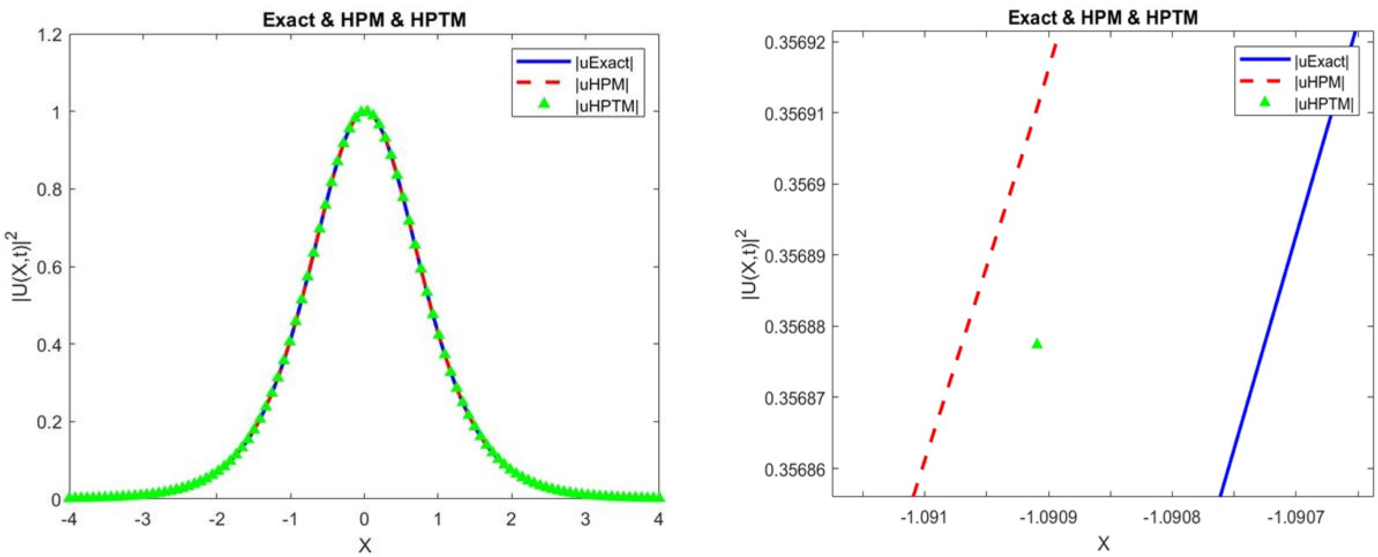
So, we can get

$$U = \sum_{m=0}^{K} U_m = U_0 + U_1 + U_2 + \cdots + U_K \tag{45}$$

$$U(X,t) = sech(X)\, e^{-Xi} + t\left(\frac{\begin{array}{c} 7\,i\,e^{-Xi}\, cosh^5(X) \\ +8\, e^{-Xi}\, cosh^3(X)\sinh(X) \\ -2\,i\,e^{-Xi}\, cosh^2(X)\, sinh^2(X) \\ +63\, cosh^6(X) - \\ 16\,i\sinh(X)\, cosh^5(X) \\ +6\,i\,e^{-Xi} cosh^2(X) - \\ 6\,i\,e^{-Xi}\, sinh^4(X) + \\ 6\,e^{-Xi}\,i \end{array}}{6\, cosh^5(X)}\right) + t^2\left(\frac{\begin{array}{c} 1032 e^{-Xi}\, cosh^4(X) - \\ 960\,i\,e^{-Xi}\sinh(X)\, cosh^3(X) \\ -3240\, e^{-Xi} cosh^2(X) \\ +e^{-Xi}1680 \end{array}}{72\, cosh^7(X)}\right). \tag{46}$$

Table 1: The absolute errors between the exact solution and approximate solutions by HPM, HPTM for U(X,t) respectively, when $X \in$ [-4,4] and t=0.01

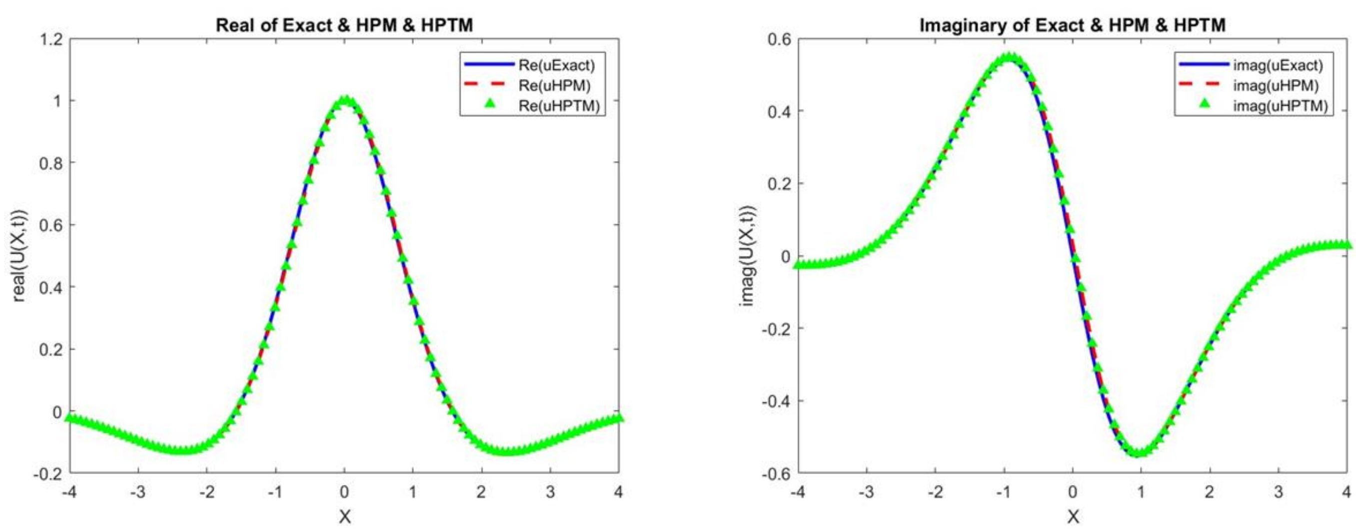
X	Exact Solution	HPM	HPTM	Absolute Error |Exact-HPM|	Absolute Error |Exact-HPTM|
-4	1.305687614593779e-03	1.305694107613501e-03	1.305694107461191e-03	6.493019721279941e-09	6.492867412125070e-09
-3.2	6.450450147789773e-03	6.450587480667723e-03	6.450587464300118e-03	1.373328779507785e-07	1.373165103455906e-07
-2.4	3.154516942753750e-02	3.154815196948520e-02	3.154815119298164e-02	2.982541947693651e-06	2.981765444137752e-06
-1.6	1.468686431534473e-01	1.469134897847851e-01	1.469139360725918e-01	4.484663133777489e-05	4.529291914459610e-05
-0.8	5.491884863153710e-01	5.491131048410096e-01	5.492296088036160e-01	7.538147436136899e-05	4.112248824506004e-05
0	9.998222432900585e-01	9.984460807118054e-01	9.997115282118056e-01	1.376162578253126e-03	1.107150782528876e-04
0.8	5.689860006351928e-01	5.689240537532594e-01	5.690447578043774e-01	6.194688193339459e-05	5.875716918457563e-05
1.6	1.542683812584001e-01	1.543146047037539e-01	1.543150734634278e-01	4.622344535387679e-05	4.669220502773186e-05
2.4	3.324429013803103e-02	3.324730898962292e-02	3.324730817127071e-02	3.018851591894689e-06	3.018033239682305e-06
3.2	6.802609526142632e-03	6.802740550956321e-03	6.802740533695097e-03	1.310248136888667e-07	1.310075524642990e-07
4	1.377165454304141e-03	1.377170078683204e-03	1.377170078522558e-03	4.624379063055198e-09	4.624218416602460e-09
Total →				1.610841879869554e-03	3.088590996873099e-04



(a) The curves of the Exact solution, HPM and HPTM.

(b) The zoomed curves indicate that the errors of the HPM, HPTM curves are close to the exact solution curve.

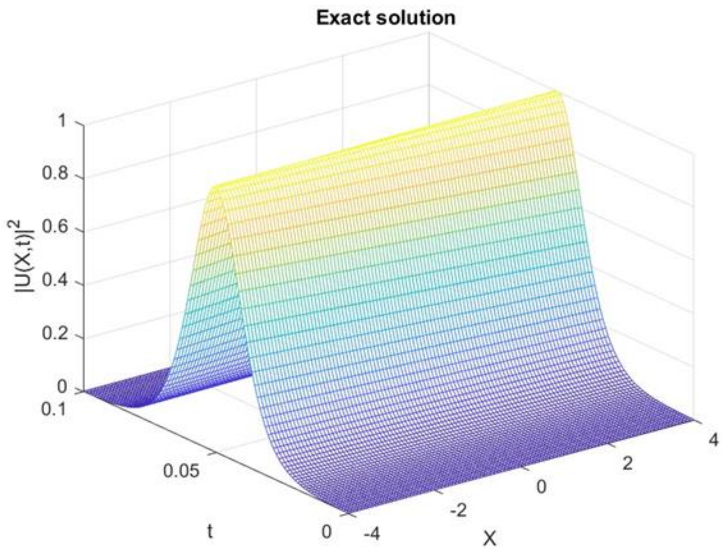
Figure 1: The comparison of HPM and HPTM using the exact solution for U(X,t) when $X \in$ [-4,4] and $t \in$ [0,0.1].



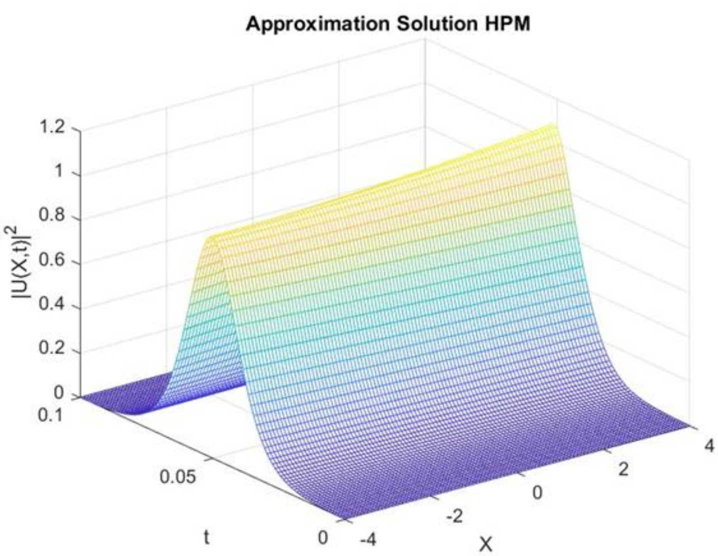
(a) Real of CQNLSE

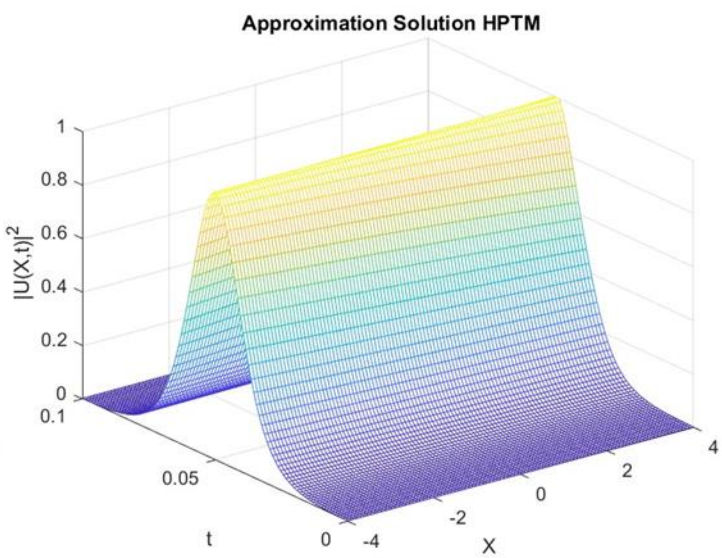
(b) Imaginary of CQNLSE

Figure 2: The curves of exact solution, HPM, and HPTM for real and imaginary, when $X \in$ [-4,4] and $t \in$ [0,0.1].



(a) Exact solution of CQNLSE

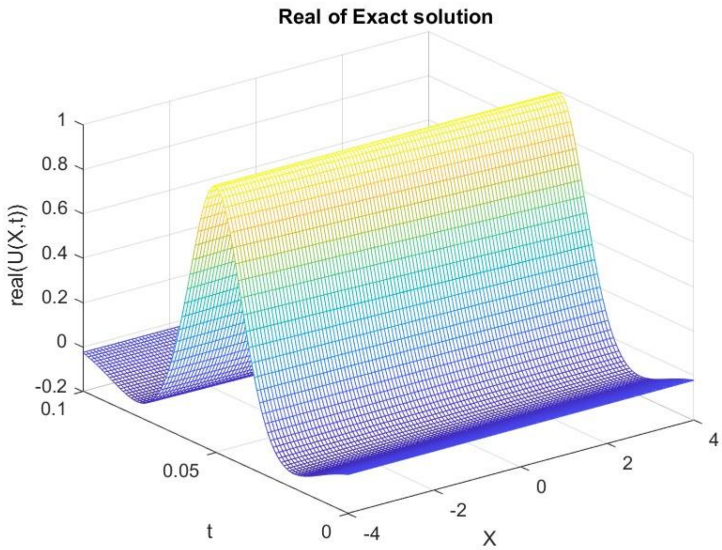




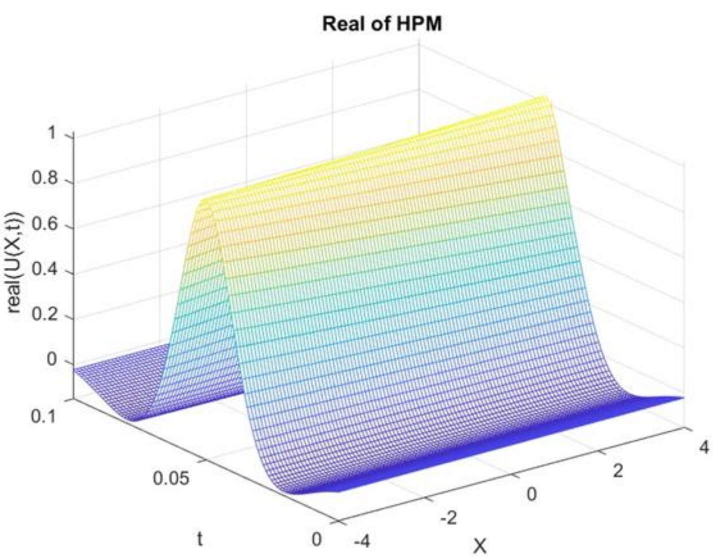
(b) HPM of CQNLSE

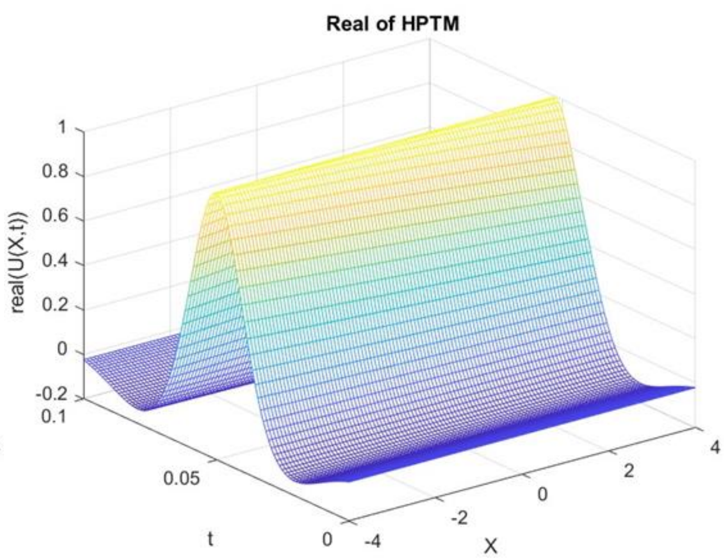
(c) HPTM of CQNLSE

Figure 3: The surfaces of the exact solution and HPM and HPTM for CQNLSE, when $X \in [-4,4]$ and $t \in [0,0.1]$.



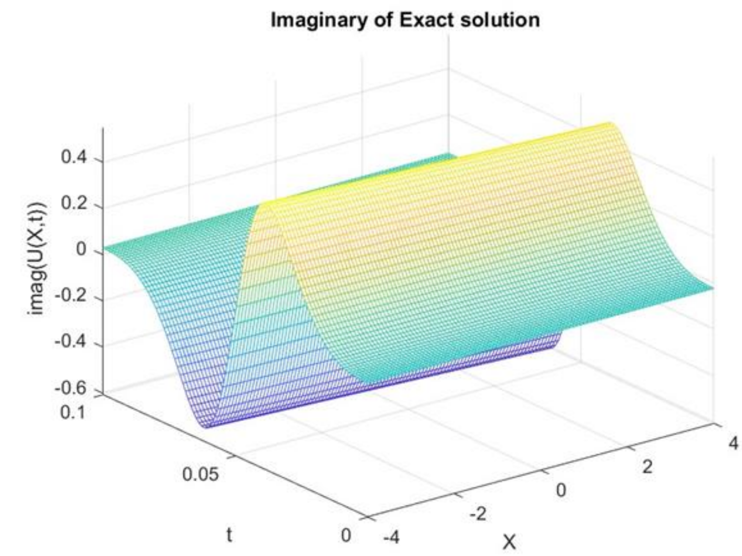
(a) Real exact solution of CQNLSE



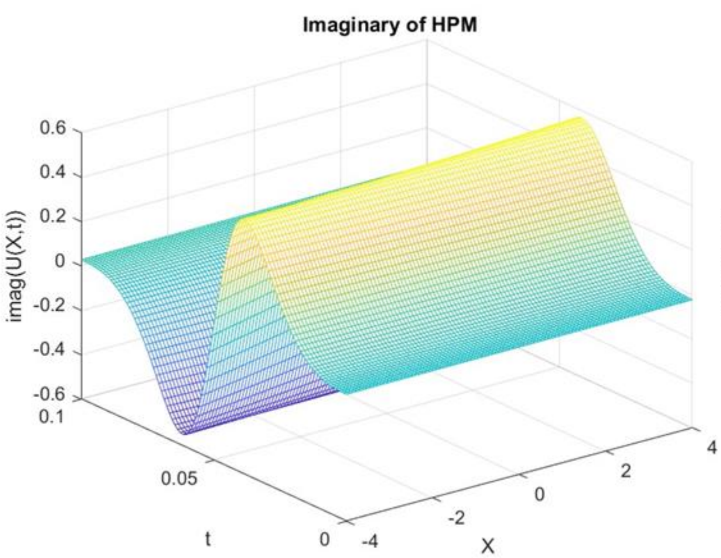


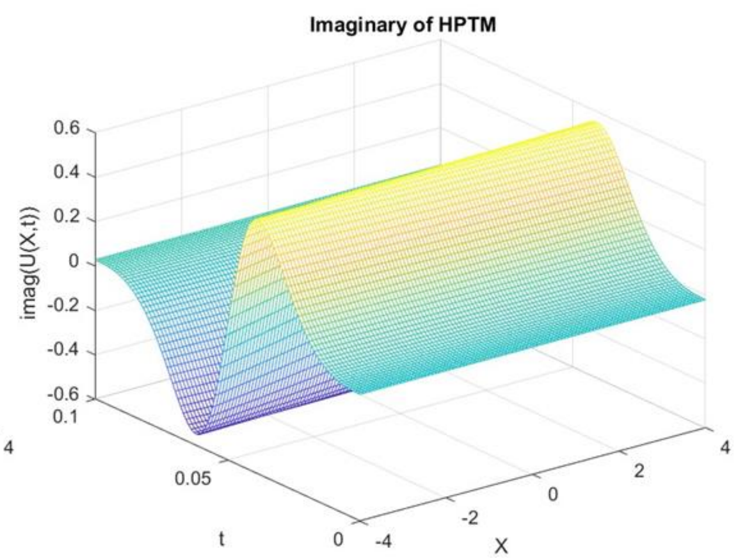
(b) Real HPM of CQNLSE (c) Real HPTM of CQNLSE

Figure 4: The surfaces of the real parts of the exact solution, HPM, and HPTM for CQNLSE, when $X \in [-4,4]$ and $t \in [0,0.1]$.



(a) Imaginary exact solution of CQNLSE





(b) Imaginary HPM of CQNLSE

(c) Imaginary HPTM of CQNLSE

Figure 5: The surfaces of the imaginary exact solution, imaginary HPM, and imaginary HPTM for CQNLSE, when $X \in [-4,4]$ and $t \in [0,0.1]$.

6 Conclusions

The HPTM and HPM are both utilized in this paper. To obtain approximate analytical solutions for the nonlinear cubic-quintic Schrödinger equation. In this study, the numerical solution of this equation was obtained for the first time by both numerical methods, HPM and HPTM. Both methods for computing series or exact solutions are significant in applied sciences due to their direct application to all differential and integral equations, reducing computational work while maintaining high numerical accuracy.

Both HPM and HPTM are effective methods for obtaining exact or series solutions to nonlinear problems. The HPM in particular requires the evaluation of the coupling of the homotopy method with classical perturbation methods. This method can take advantage of the conventional perturbation while deleting its restrictions. On the other hand, HPTM enhances this process by combining the Laplace transform with HPM, enabling more efficient computation of exact solutions for nonlinear equations.

In this work, the numerical results obtained are compared with the exact solution, while choosing from both methods. Compared to the HPM, the HPTM demonstrates significantly greater accuracy and efficiency. The numerical results confirm that the absolute error produced by HPTM is smaller than that produced by the HPM, as shown in Table 1 and Figure 1. Consequently, HPTM offers more accurate solutions that closely match the exact solutions, highlighting its superiority over HPM in solving nonlinear problems. The solutions obtained with real and imaginary parts are plotted algebraically.

Acknowledgments

To improve this work, the authors thank the editor and the referees for their insightful comments.

Author Contribution

Both authors contributed equally to this work, including the study's design and execution, data analysis, and preparation of the manuscript.

Conflict of interest

None.

References

[1] Jaradat, E. K., Alomari, O., Abudayah, M., & Al-Faqih, A. A. M. An approximate analytical solution of the nonlinear Schrödinger equation with harmonic oscillator using homotopy perturbation method and Laplace–Adomian decomposition method. *Adv. Math. Phys.* **2018**, 6765021, doi:10.1155/2018/6765021 (2018).

[2] Murad, M. A. S., Hamasalh, F. K., & Ismael, H. F. Various optical soliton solutions for the time fractional Schrödinger equation with second-order spatiotemporal and group velocity dispersion coefficients. *Opt. Quantum Electron.* **55**, 607, doi:10.1007/s11082-023-04845-2, (2023).

[3] Abdullaev, F. K., Gammal, A., Tomio, L., & Frederico, T. Stability of trapped Bose-Einstein condensates. *Phys. Rev.* **63**, 043604, doi:10.1103/PhysRevA.63.043604, (2001).

[4] Kartavenko, V. G. Soliton-like solutions in nuclear hydrodynamics. *Sov. J. Nucl. Phys.* **40**, 198–202 (1984).

[5] Zhou, C., & He, X. T. Stochastic diffusion of electrons in evolutive Langmuir fields. *Phys. Scr.* **50**, 415, doi:10.1088/0031-8949/50/4/015, (1994).

[6] Barashenkov, I. V., & Makhankov, V. G. Soliton-like "bubbles" in a system of interacting bosons. *Phys. Lett. A* **128**, 52–56, doi:10.1016/0375-9601(88)91042-0, (1988).

[7] Zhan, C., Zhang, D., Zhu, D., Wang, D., Li, Y., Li, D., ... & Nie, Y. Third-and fifth-order optical nonlinearities in a new stilbazolium derivative. *J. Opt. Soc. Am. B*. **19**, 369–375 (2002).

[8] Murad, M. A. S., Hamasalh, F. K., Malik, S., Arnous, A. H., & Iqbal, M. Analysis of soliton solutions to the nonlinear conformable Schrödinger equation in weakly non-local media using two analytic algorithms. *Nonlinear Dyn.* **113**, 11881-11892, doi:10.1007/s11071-024-10551-9, (2025).

[9] Muhsen, A. A. & Al-Hawasy, J. A. A. Solvability for couple elliptic partial differential equations. *Passer J. Basic Appl. Sci.* **7**, 452–461, doi:10.24271/psr.2025.512173.2014, (2025).

[10] Jwamer, K. H. F., Hilmi, H. D., & Hussein, S. O. Determine the fractional order T. Regge problem by applying Rishi transformation method. *Passer J. Basic Appl. Sci.* **6**, 536–542 , 10.24271/psr.2024.452507.1569, (2024).

[11] Belmonte-Beitia, J. & Cuevas, J. Solitons for the cubic–quintic nonlinear Schrödinger equation with time- and space-modulated coefficients. *J. Phys. A: Math. Theor.* **42**, 165201, doi:10.1088/1751-8113/42/16/165201, (2009).

[12] Serkin, V. N., Belyaeva, T. L., Alexandrov, I. V. & Melchor, G. M. Novel topological quasi-soliton solutions for the nonlinear cubic–quintic Schrödinger equation model. *Proc. SPIE* **4271**, 292–302, doi:10.1117/12.424706, (2001).

[13] He, J. R., Yi, L., & Li, H. M. Self-similar propagation and asymptotic optical waves in nonlinear waveguides. *Phys. Rev. E* **90**, 013202, doi: 10.1103/PhysRevE.90.013202 (2014).

[14] Caplan, R. M., Carretero-González, R., Kevrekidis, P. G. & Malomed, B. A. Existence, stability, and scattering of bright vortices in the cubic–quintic nonlinear Schrödinger equation. *Math. Comput. Simul.* **82**, 1150–1171, doi:10.1016/j.matcom.2010.11.019, (2012).

[15] Ali, M. Y., Hafez, M. G., Chowdury, M. K. H. & Akter, M. T. Analytical and traveling wave solutions to the fifth-order standard Sawada–Kotera equation via the generalized exp(−Φ(ξ))-expansion method. *J. Appl. Math. Phys.* **4**, 262–271, doi:10.4236/jamp. 2016.42033,(2016).

[16] He, J. H. Homotopy perturbation technique. *Comput. Methods Appl. Mech. Eng.* **178**, 257–262, doi.org/10.1016/S0045-7825(99)00018-3, (1999).

[17] Elbadri, M. Comparison between the homotopy perturbation method and the homotopy perturbation transform method. *Appl. Math.* **9**, 130–137, doi:10.4236/am.2018.92009, (2018).

[18] Beléndez, A., Hernandez, A., Beléndez, T., Fernández, E., Alvarez, M. L., & Neipp, C. Application of He's homotopy perturbation method to the Duffing-harmonic oscillator. *Int. J. Nonlinear Sci. Numer. Simul.* **8**, 79–88 doi:10.1515/IJNSNS.2007.8.1.79, (2007).

[19] Ariel, P. D., Hayat, T., & Asghar, S. Homotopy perturbation method and axisymmetric flow over a stretching sheet. *Int. J. Nonlinear Sci. Numer. Simul.* **7**, 399–406, doi:10.1515/IJNSNS.2006.7.4.399, (2006).

[20] Cveticanin, L. Homotopy–perturbation method for pure nonlinear differential equations. *Chaos Solitons Fractals* **30**, 1221–1230, doi: 10.1016/j.chaos.2005.08.180, (2006).

[21] Öziş, T. & Yıldırım, A. Determination of limit cycles by a modified straightforward expansion for nonlinear oscillators. *Chaos Solitons Fractals* **32**, 445–448, doi: 10.1016/j.chaos.2006.06.034, (2007).

[22] Ghorbani, A. Beyond Adomian polynomials: He polynomials. *Chaos Solitons Fractals* **39**, 1486–1492, doi: 10.1016/j.chaos.2007.06.034, (2009).

[23] Jaradat, E. K., Alomari, O., Abudayah, M., & Al-Faqih, A. A. M. An approximate analytical solution of the nonlinear Schrödinger equation with harmonic oscillator using homotopy perturbation

method and Laplace–Adomian decomposition method. *Adv. Math. Phys.* **2018**, 6765021, doi:10.1155/2018/6765021, (2018).
[24] Easif, F. H., Manaa, S. A. & Yousif, M. A. The homotopy perturbation method for solving the Kuramoto–Sivashinsky equation. *IOSR J. Eng.* **3**, 22–27 (2013).
[25] Yıldırım, A. Analytical approach to fractional partial differential equations in fluid mechanics using the homotopy perturbation method. *Int. J. Numer. Methods Heat Fluid Flow* **20**, 186–200, doi:10.1108/09615531011016957, (2010).
[26] Abbasbandy, S. Numerical solutions of the integral equations: Homotopy perturbation method and Adomian decomposition method. *Appl. Math. Comput.* **173**, 493–500, doi.org/10.1016/j.amc.2005.04.077, (2006).
[27] Mohyud-Din, S. T. & Noor, M. A. Homotopy perturbation method for solving partial differential equations. *Z. Naturforsch. A* **64**, 157–170, doi:10.1515/zna-2009-3-402, (2009).
[28] Sweilam, N. H. & Khader, M. M. Exact solutions of some coupled nonlinear partial differential equations using the homotopy perturbation method. *Comput. Math. Appl.* **58**, 2134–2141, doi: 10.1016/j.camwa.(2009).03.059, (2009).
[29] Singh, J., Kumar, D. & Rathore, S. Application of the homotopy perturbation transform method for solving linear and nonlinear Klein–Gordon equations. *J. Inf. Comput. Sci.* **7**, 131–139 (2012).
[30] Lai, X. J., Zhang, J. F., & Luo, J. F. Adomian decomposition method for approximating the solution of the high-order dispersive cubic–quintic nonlinear Schrödinger equation. *Z. Naturforsch. A* **61**, 205–215, doi.org/10.1515/zna-2006-5-601, (2006).
[31] Sadighi, A. & Ganji, D. D. Analytic treatment of linear and nonlinear Schrödinger equations: a study with homotopy perturbation and Adomian decomposition methods. *Phys. Lett. A* **372**, 465–469, doi:10.1016/j.physleta.2007.07.065, (2008).
[32] He, J. H. A coupling method of a homotopy technique and a perturbation technique for non-linear problems. *Int. J. Non-Linear Mech.* **35**, 37–43, doi:10.1016/S0020-7462(98)00085-7,(2000).
[33] Sharma, D., Singh, P. & Chauhan, S. Homotopy perturbation transform method with He's polynomials for the solution of coupled nonlinear partial differential equations. *Nonlinear Eng.* **5**, 17–23, doi: 10.1515/nleng-2015-0029, (2016).
[34] Jassim, H. K. Homotopy perturbation algorithm using Laplace transform for the Newell–Whitehead–Segel equation. *Int. J. Adv. Appl. Math. Mech.* **2**, 8–12 (2015).