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Original Article

Soft Set Representations of First Barycentric Refinements of Simple Graphs

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ABSTRACT

We investigate soft set representations of the first barycentric refinements of simple graphs. Simple graphs under consideration are those with cliques of size at most three. In such representations, we use soft sets to describe the first barycentric refinements of such simple graphs. We use the soft set representation of two graph operations to obtain the soft set representation of the first barycentric refinements of the graphs under consideration. Specifically, we deal with vertex-insertion and spider-insertion operations in terms of soft sets. Furthermore, we illustrate and investigate some properties of soft set representations of the first barycentric refinements of some special simple graphs.

Keywords: Simple graphs, Graph operations, Soft sets, Soft set representations.

1 Introduction

In graph theory, the topic of graph representations was fundamentally established to facilitate understanding certain properties of graphs pictorially. The classical visualization of a graph is all about using two simple geometric objects: small circles or discs and simple curves usually line segments. Circles or discs represent the vertices and line segments illustrate the edges of the graph under consideration. The method of coupling the circles among each other using line segments provides all needed information to describe such a graph. However, the interaction between graph theory and many other disciplines boosted investigators and researchers to introduce variety of representations of graphs using different objects and mathematical structures. For example, topological graphs can be represented in various types such as polygon representations of topological graphs [1]. Adjacent matrices and incident matrices of graphs are two other classical methods of representing graphs which they serve very well in computer sciences. Since graphs basically describe the relationships between sets and their elements. Such a virtue naturally motivated some researchers to find possible ways to represent graphs using sets. Soft sets can be used to represent graphs by considering the neighborhoods of the vertices of the graphs. This article mainly depends on representing graphs using soft sets. Soft set theory [2] appeared as a new tool to resolve difficulties confronted by other related theories such as fuzzy set theory [3-4] and rough set theory [5] within the context of uncertainty of data. Soft set theory, in comparison with other similar theories, can treat such difficulties directly [6].

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In this article, we study representing the first barycentric refinement of simple graphs with cliques of sizes not exceeding three using soft set theory. In [7], a new method of representing simple graphs using soft sets was introduced. Such a representation depends on the concept of neighboring of vertices. The first barycentric refinement of a graph, say G, is a graph derived from G where its vertices are the elements in the simplicial complex of the graph G. Each edge of the first barycentric refinement of G is linking two vertices u and v whenever u contains v or v contains u. In a similar manner, one can define the second barycentric refinement applied on the first barycentric refinement and so on. The major aim of this article is to demonstrate the possibility of describing the first barycentric refinements of simple graphs using soft sets. For this purpose, we deal with two graph operations to find the first barycentric refinements of simple graphs, and we consider the soft set representations of such operations and examine further properties. We also consider soft set representations of some special well-known graphs. Therefore, this work can significantly contribute to a profound understanding of barycentric refinement of simple graphs and eventually can ease digesting other related topics with barycentric refinement. For instance, this soft set representation can be used to define the notion of dual cells [8].

Soft set representation can be contributed to studying some aspects of the graph spectrum [9].

The rest of the sections of this article includes the following: Section 2 includes the fundamental ingredient that we need within this article. We briefly recall necessary concepts in graph theory, barycentric refinement, and soft set theory. Section 3 includes soft set representations of simple graphs and represents the first barycentric refinements of simple graphs using soft sets. In section 4, we explore soft set representations of some known simple graphs.

2 Preliminaries

In this section we recall the required concepts that we need to describe and investigate the possibility of representing the first barycentric refinements of certain simple graphs using soft sets.

2.1 Graph theory: Basic concepts

A graph $G = (V, E)$ consists of a finite set of vertices V and a set of edges E where edges are unordered pairs of two vertices [10]. We assume that $G = (V, E)$ is a simple graph which means G has neither multi-edges nor loops. Moreover, we assume that both V and E are finite.

We note that the barycentric refinement of graphs with complete subgraphs of order greater than three complicates the process of computing the soft set representation of such barycentric refinement. For the same reason, we only dealt with the first refinement and avoided further refinement iterations. Therefore, we assert that this article deals with only simple graphs of cliques of size at most three.

An edge e_{uv} is an unordered pair $\{u, v\}$ which is incident to both u and v. In this case, u and v are called the endpoints of e_{uv}. Conveniently, if $u, v \in V$ are adjacent, then $e_{uv} = e_{vu}$. The degree of v in V, $deg(v)$, is the cardinality of the set $\{e_{uv} \in E : u \in V\}$, that is $deg(v) = |\{e_{uv} \in E : u \in V\}|$. The neighborhood of vertex $v \in V$ is the set $N_G(v) = \{u \in V : e_{uv} \in E\}$.

A walk W is a sequence $v_1, e_{v_1v_2}, v_2, e_{v_2v_3}, \dots, e_{v_{k-1}v_k}, v_k$ where $i = 1, 2, \dots, k$, the vertex v_1 is called the initial vertex of W and v_k is called the terminal vertex of W. A walk is simple if no vertex in that walk is repeated. The walk is closed if $v_k = v_1$. A simple closed walk is called a cycle if the vertex $v_k = v_1$ is the only repeated vertex. This cycle we call it a k-cycle. Hence, a k-cycle has k vertices and k edges. A path P in G is a walk so that all the vertices of this walk are distinct. If v_1 and v_k are the initial and terminal vertices of P then P is called v_1v_k-path.

Operations on graphs play many significant roles in graph theory. One advantage of graph operations is to enlarge given graphs for certain reasons. Also, operations on graphs can be used to build various classes of graphs from some small graphs. Such operations preserve certain properties of the class of graphs. Common and fundamental operations on graphs are insertion and deletion [11]. Specifically, insertion and deletion operations can involve inserting vertices and/or edges [12]. In the following, we describe two kinds of insertions. Such operations will be used to obtain the first barycentric refinements of simple graphs with cliques of size at most three.

Let $G = (V, E)$ be a simple graph and $e_{uw} \in E$. Let $G_{w,e_{uv}} = (V', E')$ be the graph obtained from G by removing the edge e_{uv} and adding a vertex w and two edges e_{uw} and e_{vw} where $V' = V \cup \{w\}$ and $E' = (E - \{e_{uv}\}) \cup \{e_{uw}, e_{vw}\}$. The operation of obtaining $G_{w,e_{uv}}$ from G is called a vertex-inserting operation. The graph $G_{w,e_{uv}} = (V', E')$ is obtained by a vertex-inserting operation on G. We occasionally call the previous operation a subdivision operation. The vertex w is called a subdivision vertex of the edge e_{uv}.

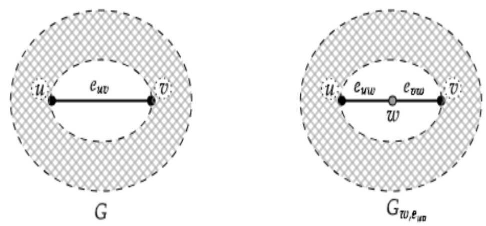

Figure 1: A subdivision operation. G is a graph which has an edge e_{uv} adjacent to two vertices u and v. $G_{w,e_{uv}}$ is the graph that is obtained from G by deleting the edge e_{uv} and adding a new vertex w and two new edges e_{uw} and e_{vw}.

Let $G = (V, E)$ be a simple graph. A subdivision of a graph G, denoted by $S(G) = (V_S, E_S)$, is obtained from G by performing a subdivision operation on each $e \in E$.

Example 1. Consider the graph G in Figure 2, The subdivision of G is the graph $S(G)$. Since there are exactly 14 edges in G. Thus, $S(G)$ is obtained from G by performing 14 vertex-insertion operations. Furthermore, the number of vertices in $S(G)$ is twice of G.

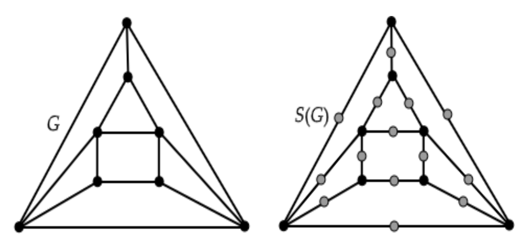

Figure 2: A simple graph G and its subdivision $S(G)$.

Lemma 1. Let $S(G) = (V_S, E_S)$ be the subdivision of a simple graph $G = (V, E)$. Then,

1. $S(G)$ is obtained from G by performing exactly $|E|$ subdivision operations.

2. $|V_S| = |V| + |E|$.

3. $|E_S| = 2|E|$.

Proof. (1) follows from the definition of $S(G)$.

(2) Since we need to perform exactly $|E|$ edge subdivision. In each of such operation, we insert a new vertex. Thus, the total number of vertices of $S(G)$ will be equal to the number of vertices of G plus $|E|$.

(3) For each subdivision operation, we replace an edge of G with two new edges. Hence the number of the edges in $S(G)$ is the double number of the edges of G.

Definition 1. Let $G = (V, E)$ be a simple graph and $S = v_1, e_{v_1v_2}, v_2, e_{v_2v_3}, v_3, e_{v_3v_4}, v_4, e_{v_4v_5}, v_5, e_{v_5v_6}, v_6, e_{v_6v_1}, v_1$ be a 6-cycle in G. Let $G_{w,S} = (V_w, E_w)$ be the graph obtained from G by adding a new vertex w of degree 6 with 6 edges $e_{wv_i}$where $i \in \{1, 2, \dots, 6\}$. Thus, $V_w = V \cup \{w\}$ and $E_w = E \cup \{e_{wv_1}, \dots, e_{wv_6}\}$. The operation of obtaining $G_{w,S}$ from G is called a spider-inserting operation.

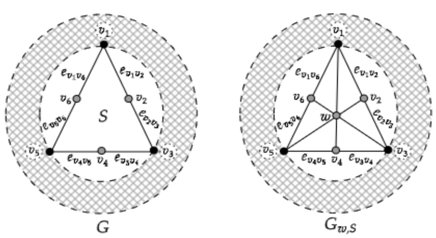

Figure 3: A Spider insertion operation: G is a graph which has a 6-cycle S. $G_{w,S}$ is the graph that is obtained from G by inserting a new vertex w and six new edges such that w is adjacent to each vertex in S. We say that w is a spider-hup vertex.

2.2 First Barycentric Refinementcs of Simple Graphs

In the following we recall the concept of barycentric refinements of simple graphs and present examples of extracting them for some simple graphs.

Definition 2. [9] Let $G = (V, E)$ be a finite simple graph, the first barycentric refinement of G is the graph $G_1 = (V_1, E_1)$ for which V_1 consists of all non-empty complete subgraphs of G and E_1 consists of all unordered distinct pairs in V_1, for which one is a subgraph of the other.

Example 2. Consider the complete graph K_2 in Figure 4. There are two complete subgraphs of G, one consists of the vertex u and the other consists of v, and one complete subgraph K_2 itself. Thus, in the first barycentric refinement $G_1 = (V_1, E_1)$, V_1 is consists of three vertices, two vertices stand for the complete subgraphs of the vertices u and v. We keep the same names of these two vertices. The third vertex stands for the complete subgraph K_2, here we add a new vertex, say w. Thus, $V_1 = \{u, v, w\}$. Now, it is clear that the complete subgraph that consists of the vertex u is a subgraph of K_2. Hence, there is an edge e_{uw}. For the same reason, the inclusion relation between K_1 that consists of v and K_2, there is an edge e_{vw}. Thus, $E_1 = \{e_{vw}, e_{uw}\}$. See Figure 4.

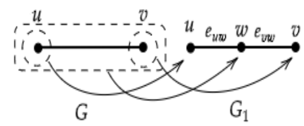

Figure 4: Building the first barycentric refinement of a complete graph $G = K_2$.

Example 3. Consider the complete graph $K_3 = G = (V, E)$ in Figure 5. There are three complete subgraphs K_1 of G, each one consists of one vertex; u, v and x. G also contains three complete subgraphs K_2 and K_3 itself. Thus, in the first barycentric refinement $G_1 = (V_1, E_1)$, V_1 is consists of seven vertices, three vertices stand for the complete subgraphs of the vertices u, v and x. We keep the same names of these three vertices. We add three vertices a_1, a_2 and a_3 which stand for the three complete subgraphs K_2 in G. Finally, we add a vertex w that stands for the complete graph K_3 itself. Thus, $V_1 = \{u, v, x, a_1, a_2, a_3, w\}$. Now, it is clear that the complete subgraph that consists of only the vertex u is a subgraph of two K_2 subgraphs. But each K_2 generates a new vertex a_i, each a_i is an end vertex of two edges. Now, G itself contains all subgraphs of type K_1 and K_2 . See the labelled edges of the graph G_1 in Figure 5.

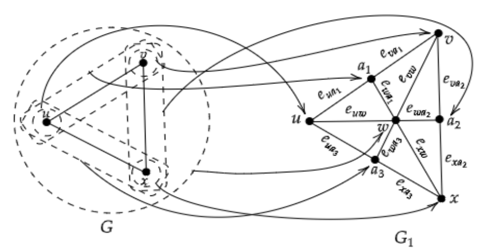

Figure 5: Building the first barycentric refinement of the complete graph $G = K_3$.

We observe that if G is a simple graph with no complete subgraphs other than K_1 and K_2, then the first barycentric refinement of G, G_1, is exactly the subdivision graph of G. That is, $G_1 = S(G)$. If G has r complete subgraphs K_3, then G_1 can be obtain from $S(G)$ after performing r spider-insertion operations.

In comparison with building the first barycentric refinement of K_3 in Example 3, we note the first barycentric refinement of the complete graph K_3 can be found by performing a spider-insertion operation on the subdivision graph $S(K_3)$ of K_3.

Lemma 2. Let $G = (V, E)$ be a simple graph such that G has no 3-cycle, then the first barycentric refinement, G_1, of G is the graph $S(G)$.

Proof. This follows from being G contains no complete subgraph other than those with one and two vertices. Hence, the first barycentric refinement of G obtained by performing $|E|$ vertex-insertion operations on G. Thus, the resulting graph after obtaining all the mentioned operations on G is $S(G)$.

Lemma 3. Let $G_1 = (V_1, E_1)$ be the first barycentric refinement of a simple graph $G = (V, E)$. If $u \in N_G(v)$ then $u \notin N_{G_1}(v)$.

Proof. let $u \in N_G(v)$ and $e_{uv} \in E$ be the edge incident to u and v. Consider the graph $G_{w,e_{uv}}$ which is obtained from G by a subdivision operation on edge e_{uv}. Since G is simple, so u and v are adjacent to each other via only the edge e_{uv}. However, in $G_{w,e_{uv}}$, a new vertex w and two new edges are added to this graph while the edge e_{uv} is deleted. Hence the neighborhoods of u and v are differed in such a way that they are no longer adjacent to each other.

Let e_{uv} be an edge in a simple graph G. Consider the subgraph $T_{e_{uv}}$ corresponding to the union of all the 3-cycles of G incident to e_{uv}. That is, $T_{e_{uv}}$ is the collection of all 3-cycles that contain the edge e_{uv}. Hence, an element of $T_{e_{uv}}$ is a 3-cycle that contains e_{uv}.

2.3 Soft Set Theory: Basic Concepts

In the following, we explore some basic concepts in soft set theory which we use in the next section.

Definition 3. [2] Let $A \subset E$ where E stands for a set of parameters. A soft set, (F, A), on the universe set X consists of the set of ordered pairs $(F, A) = \{(e, F(e)) : e \in E, F(e) \in P(X)\}$ where the function $F : E \to P(X)$ is defined by $F(e) = \emptyset$ if $e \notin A$. The function F is called an approximate function of (F, A).

Example 4. Consider the universe $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the decision parameters sets. Now, let $A = \{e_2, e_3, e_5\}$ and $F(e_2) = \{x_2, x_6\}$, $F(e_3) = \{x_1, x_3, x_4\}$ and $F(e_5) = \{x_1, x_7, x_8\}$.

Therefore, our soft set is $(F, A) = \{(e_2, \{x_2, x_6\}), (e_3, \{x_1, x_3, x_4\}), (e_5, \{x_1, x_7, x_8\})\}$.

In the following we recall some basic concepts of soft set theory [13-14]. If $F(e) = \emptyset\ \forall e \in A$, then such a soft set with this approximate function is called an empty soft set. An A-universe set is a soft set such that $F(e) = X$ for all $e \in A$. In case that $A = E$, then (F, A) is called a universe soft set. Now, suppose that (F, A) and (G, B) are soft sets. (G, B) is a soft subset of (F, A) whenever $G(e) \subseteq F(e)$ for all $e \in B$. Such an operation is denoted by $(G, B) \tilde{\subseteq} (F, A)$. If it happens that $(G, B) \neq (F, A)$, then (G, B) is a soft proper subset of (F, A). We denote such implication as $(G, B) \tilde{\subsetneq} (F, A)$. Finally, let (F, A) be a soft set on the universe set X. The soft complement of (F, A), is defined by $F^c(e) = X - F(e)$ for all $e \in A$. Such an operation is denoted by $(F, A)^{\tilde{c}}$.

3 Soft Set Representations of First Barycentric Refinements of Simple Graphs

In this section, first we recall the representation of simple graphs using soft sets. Then, we study the *SSR* of the first barycentric refinements of simple graphs. In [7], representing simple graphs using soft sets was introduced. In the following, we recall the definition of such representation and declare some related concepts.

Definition 4. [7] Let $G = (V, E)$ be a simple graph. Define the function $F_G : V \to P(V)$ such that

$$F_G(v) = \begin{Bmatrix} \{u \in V\} & \text{if } e_{uv} \in E \\ \phi & \text{if } e_{uv} \notin E \end{Bmatrix}$$

for every $v \in V$. The function F_G defines a soft set (F_G, V) over the power set $P(V)$. The soft set (F_G, V) is called the soft set representation of G, in short *SSR* of G.

Given the *SSR* of a simple graph $G = (V, E)$, (F_G, V). The image of $v \in V$ under the function F_G, i.e. $F_G(v)$, is called F_G-image of v. One can observe that $u \in F_G(v)$ if and only if $v \in F_G(u)$ for $u, v \in V$. Clearly, if $v \in V$ is an isolated, then its F_G-image is the empty set, i.e. $F_G(v) = \emptyset$. We denote the family of all F_G-images of an *SSR* of G by $\mathcal{F}_G$. Thus, it is clear that $|\mathcal{F}_G| = |V|$. Now, if V is partitioned into two sets M and N. Then we denote $\mathcal{F}_{G,M}$ and $\mathcal{F}_{G,N}$ the collection of F_G-images of all $u \in M$ and $w \in N$, respectively.

Lemma 4. Let $G = (V, E)$ be a simple graph and e_{uv} be an edge in G. Suppose that $G_{w,e_{uv}} = (V', E')$ be the simple graph that obtained from G by a subdivision operation. If (F_G, V) is the *SSR* of G. Then $\left(F_{G_{w,e_{uv}}}, V'\right) = [(F_G, V) - \{(u, \{v\}), (v, \{u\})] \tilde{\cup} \{(w, \{u, v\})\} \tilde{\cup} \{(v, \{w\})\} \tilde{\cup} \{(u, \{w\})\}$ is the *SSR* of $G_{w,e_{uv}}$.

Proof: Consider the graph $G = (V, E)$ and its *SSR* (F_G, V). In $G_{w,e_{uv}} = (V', E')$, we notice that $V' = \mathrm{V} \cup \{\mathrm{w}\}$ and $E' = (\mathrm{E} - \{e_{uv}\}) \cup \{e_{uw}, e_{vw}\}$. Thus, in $G_{w,e_{uv}}$, a new vertex w and two new edges are added to this graph while the edge e_{uv} is deleted. Hence the neighborhoods of u and v are differed in such a way that they are no longer adjacent to each other.

Lemma 5. Let $G = (V, E)$ be a simple graph and $S = v_1, e_{v_1v_2}, v_2, e_{v_2v_3}, \dots, v_6, e_{v_6v_1}, v_1$ be a 6-cycle in G. Suppose that $G_{S,w} = (V', E')$ be the simple graph that obtained from G by a spider-insertion operation (via) S. If (F_G, V) is the *SSR* of G. Then,

$\left(F_{G_{S,w}}, V'\right) = (F_G, V) \tilde{\cup} \{(\mathrm{w}, \mathrm{V(S)})\} \tilde{\cup} \{(v_i, \{\mathrm{w}\}) : \mathrm{i} = 1, 2, \dots, 6\}$ is the *SSR* of $G_{S,w}$.

Example 5. Consider the simple graph G in Figure 6, It is clear that the soft set representation (*SSR*) of G is $(F_G, V) = \{(v_1, \{v_2, v_6\}), (v_2, \{v_1, v_3\}), (v_3, \{v_2, v_4, v_{11}\}), (v_4, \{v_3, v_5\}), (v_5, \{v_4, v_6, v_7\}), (v_6, \{v_1, v_5\}), (v_7, \{v_5, v_8\}), (v_8, \{v_7, v_9\}), (v_9, \{v_8, v_{10}\}), (v_{10}, \{v_9, v_{11}\}), (v_{11}, \{v_3, v_{10}\})\}$.

In Figure 6, we can easily depict the *SSR* of $G_{e,w} = (V', E')$ as in the following:

$(F_{G_{S,w}}, V') = (F_G, V) \tilde{\cup} \{(w, \{v_1, v_2, \ldots, v_6\})\} \tilde{\cup} (v_1, \{w\})$
$\tilde{\cup} (v_2, \{w\}) \tilde{\cup} (v_3, \{w\}) \tilde{\cup} (v_4, \{w\})$
$\tilde{\cup} (v_5, \{w\}) \tilde{\cup} (v_6, \{w\} = \{(v_1, \{v_2, v_6, w\}), (v_2, \{v_1, v_3, w\}),$
$(v_3, \{v_2, v_4, v_{11}, w\}), (v_4, \{v_3, v_5, w\}),$
$(v_5, \{v_4, v_6, v_7, w\}), (v_6, \{v_1, v_5, w\}), (v_7, \{v_5, v_8\}), (v_8, \{v_7, v_9\}),$
$(v_9, \{v_8, v_{10}\}), (v_{10}, \{v_9, v_{11}\}), (v_{11}, \{v_3, v_{10}\}),$
$(w, \{v_1, v_2, v_3, v_4, v_5, v_6\})\}.$

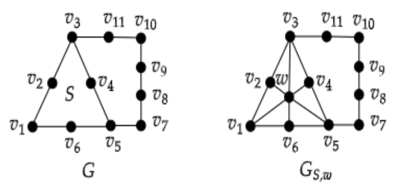

Figure 6: A graph $G = (V, E)$ and performing a spider-insertion operation on G to obtain $G_{S,w}$.

Since the first barycentric refinement, G_1, of a simple graph G is also a simple graph. Hence, it is natural to consider the soft set representation of G_1 .

Example 6. Consider the simple graph G in Figure 7.
The *SSR* of G is $(F_G, V) = \{(u, \{v, w\}), (v, \{u, w\}), (w, \{u, v\})\}$.
Now, the first barycentric refinement is the graph G_1 depicted in Figure 7. The *SSR* of G_1 is
$(F_{G_1}, V_1) = \{(u, \{v_1, w_1, x\}), (v, \{u_1, v_1, x\}), (w, \{v_1, w_1, x\}),$
$(u_1, \{u, v, x\}), (v_1, \{v, w, x\}), (w_1, \{u, w, x\}),$
$(x, \{u, v, w, u_1, v_1, w_1\})\}.$

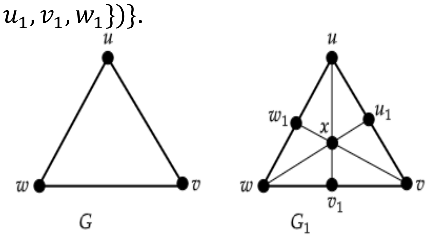

Figure 7: The graph G with its soft representation G_1 given in Example 6.

Proposition 1. Let $G_1 = (V_1, E_1)$ be the first barycentric refinement of a simple graph $G = (V, E)$. Let (F_G, V) and (F_{G_1}, V_1) be the *SSR* of G and G_1, respectively. If $v \in F_G(u)$ then $v \notin F_{G_1}(u)$.

Proof. The proof follows directly from Lemma 3.

Proposition 2. Let $G = (V, E)$ be a simple graph and $G_1 = (V_1, E_1)$ be the first Barycenteric refinement of G. If $y \in V_1$ is the subdivision of $e_{uv} \in E$, then $|F_{G_1}(y)| = 2 + |T_{e_{uv}}|$.

Proof. Let $|T_{e_{uv}}| = r$. Since y is a subdivision vertex of e_{uv} . So $\{u, v\} \subset F_{G_1}(y)$. If $G_1 = S(G)$, then $F_{G_1}(y) = 2$. Now, let $G_1 \neq S(G)$ and $T_{e_{uv}} \neq \emptyset$. Since $e_{uv} \in \cap T_{e_{uv}}$, so each element of $T_{e_{uv}}$ will contain an edge adjacent to y in G_1 as a result of a spider-insertion operation. Thus, $|F_{G_1}(y)| = 2 + |T_{e_{uv}}|$.

4 First Barycentric Refinements of Some Special Graphs and their Soft Set Representations

In this section, we explore soft set representations of the first barycentric refinements of some special graphs. For each special graph tackled in this section, we first describe how to find its first barycentric refinement and then we present its soft set representation.

4.1 Cycle Graphs

A graph $G = (V, E)$ is called a cycle graph if all its vertices $V = \{v_1, v_2, \ldots, v_{\mathrm{n}}\}$ are distinct and for every two vertices v_{i} and $v_{\mathrm{i+1}}$, $e_{v_{\mathrm{i}}v_{\mathrm{i+1}}} \in E$ for $1 \leq i \leq n$ and $e_{v_1 v_{\mathrm{n}}} \in E$. Such a graph is usually denoted by C_{n}.

Lemma 5. Let $C_n = G = (V, E)$ be a cycle graph of length $n > 3$. Then, $G_1 = S(G)$.

Proof: Since G has no 3-cycles, then it is clear that G_1 can be obtained from G by performing a subdivision operation on each edge of G. Thus, $G_1 = S(G)$.

Proposition 3. Let $G_1 = (V_1, E_1)$ be the first barycentric refinement of a cycle graph $C_n = G = (V, E)$ with $n > 3$. Let (F_G, V) and (F_{G_1}, V_1) be the *SSR* of G and G_1, respectively. Then,

1. $|F_G(v)| = |F_{G_1}(v)| = 2$ for every $v \in V \cup V_1 = V_1$.
2. $|\mathcal{F}_{G_1}| = 2|\mathcal{F}_G|$.
3. $|E_{G_1}| = 2|E_G|$.
4. $T_{e_{uv}} = \emptyset$ for each $e_{uv} \in E$.

Proof. (1) Since the only cycle in C_n is itself and $n > 3$. Thus, $G_1 = S(G)$ which means G_1 is also a cycle graph. Hence, $\deg(v) = 2$ for each $v \in V'$.

(2) From the proof of (1), G_1 is a cycle and Lemma 5 asserts that $G_1 = S(G)$. But, G_1 is obtained from G by performing vertex-inserting operations on the edges of G. Since, G has n edges, so the number of edges of G_1 is double number of edges of G. But in a cycle graph, the numbers of its vertices and edges are equal. Thus, $|V_1| = 2|V|$. From (1) we get the required.

(3) From the proof of (1), G_1 is a cycle and Lemma 5 asserts that $G_1 = S(G)$. But, G_1 is obtained from G by performing vertex-inserting operations on the edges of G. Since, G has n edges, so the number of edges of G_1 is double number of edges of G.

(4) This follows from being C_n has no cycle other than itself and $n > 3$.

Example 7. Consider the cycle graph $G = (V, E)$ which is in Figure 8. The *SSR* of G is
$(F_G, V) = \{(u, \{v, x\}), (v, \{\mathrm{u}, \mathrm{w}\}), (w, \{\mathrm{v}, \mathrm{x}\}), (x, \{\mathrm{u}, \mathrm{w}\})\}.$
$(F_{G_1}, V_1) = \{(u, \{a_1, a_4\}, (\mathrm{v}, \{a_1, a_2\}), (\mathrm{w}, \{a_2, a_3\}),$
$(\mathrm{x}, \{a_3, a_4\}), (a_1, \{u, v\}), (a_2, \{v, w\}),$
$(a_3, \{w, x\}), (a_4, \{u, x\})\}.$

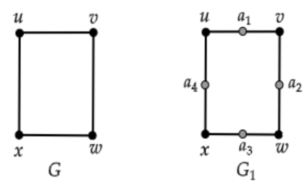

Figure 8: A cycle graph G with its first barycentric graph G_1.

4.2 Wheel Graphs

A wheel graph $W_n = (V, E)$ is a graph with $|V| = n$ that contains an $(n-1)$-cycle such that for every vertex v in this cycle is adjacent to one another vertex, say w, that is $e_{vw} \in E$. The vertex w is called the hup of W_n. Let G be the wheel graph with n vertices. The subdivision of G, $S(G)$, is obtained from G by preforming $2n-2$ vertex-insertion operations on G. Since in G there are $(n-1)$ 3-cycles, then we need to perform $n-1$ spider-insertion operations on $S(G)$ to obtain the first barycentric refinements $G_1 = (V_1, E_1)$ on the wheel graph G.

Lemma 6. Let G be the wheel graph with n vertices and let w be the hub of G. Then, there are exactly $(n-1)$ 3-cycles in G.

Lemma 7. Let G_1 be the first barycentric refinement of the wheel graph W_n, where n is a positive integer. Then G_1 is obtained from $S(W_n)$ by performing exactly $n-1$ spider-insertion operations.

Proof. Since there are $(n-1)$ 3-cycles in W_n, then we can perform exactly $(n-1)$ spider-insertion operations in $S(W_n)$ to obtain G_1.

Proposition 4. Let $W_n = G = (V, E)$ be a wheel graph with n vertices and let w be the hup of G. Assume that $G_1 = (V_1, E_1)$ be its first barycentric refinement. Then the following are held:

1. $|F_{G_1}(\mathrm{v})| = |F_G(\mathrm{v})| + 2 = 5$ for every $v \in V - \{w\}$.
2. $|F_{G_1}(u)| = 3$ for every edge-subdivision vertex u in V_1.
3. $|F_{G_1}(s)| = 6$ for every spider-hub vertex s in V_1.
4. $|F_{G_1}(w)| = 2(\mathrm{n}-1)$.
5. $|E_{G_1}| = 2|E_G| + 6(n-1) = 10(n-1)$.
6. $|\mathcal{F}_{G_1}| = 4(n-1) + 1 = 4n - 3$.

Proof. (1) In $S(W_n)$, $|F_{G_1}(\mathrm{v})| = 3$ for each $v \in V - \{w\}$. Now, each $v \in V - \{w\}$ is a part of exactly two 3-cycles. Hence, after performing spider-insertion operations in $S(W_n)$ to obtain G_1, two new edges will be adjacent to v. Thus, $|F_{G_1}(\mathrm{v})| = 5$.
(2) and (3) are obvious.
(4) Follows from Lemma 7.
(5) The subdivision of G has $2|E_G|$. By Lemma 6, G has $(n-1)$ 3-cycles. To obtain the first barycentric refinement, we need to perform $(n-1)$ spider-insertion operations in $S(W_n)$ to obtain G_1. In each such operation, 6 edges will be added as well. Hence, the number of the edges to be added will be $6(n-1)$. Therefore, the number of all edges of G_1 is $2|E_G| + 6(n-1)$.
(5) Since $|E| = 2(|V| - 1)$, thus $S(W_n)$ has $n + 2(n-1)$ vertices. By Lemma 6, there are $(n-1)$ 3-cycles. By Lemma 7, to obtain the first barycentric refinement, we need to perform $(n-1)$ spider-insertion operations in $S(W_n)$ to obtain G_1. Hence, $(n-1)$ other vertices to be introduced to the previous vertices. Consequently, $|\mathcal{F}_{G_1}| = n + 2(n-1) + (n-1)$.

Example 8. Consider the graph $G = (V, E)$ in Figure 9. The SSR of G is
(F_G, V)
$= \{(u, \{v, w, x\}), (v, \{u, w, x\}), (w, \{u, v, x\}), (x, \{u, v, w\})\}$
$(F_{G_1}, V_1) = \{(u, \{a_1, a_3, b_1, b_5, b_6\}), (v, \{a_1, a_2, b_1, b_2, b_3\}),$
$(w, \{a_2, a_3, b_3, b_4, b_5\}), (x, \{b_1, b_2, b_3, b_4, b_5, b_6\}), (a_1, \{u, v, b_1\}),$
$(a_2, \{v, w, b_3\}), (a_3, \{u, w, b_5\}), (b_1, \{u, v, x, a_1, b_2, b_6\}),$
$(b_2, \{v, x, b_1, b_3\}), (b_3, \{v, w, x, a_2, b_2, b_4\}), (b_4, \{w, x, b_3, b_5\}),$
$(b_5, \{u, w, x, a_3, b_4, b_6\}), (b_6, \{u, x, b_1, b_5\})\}$.

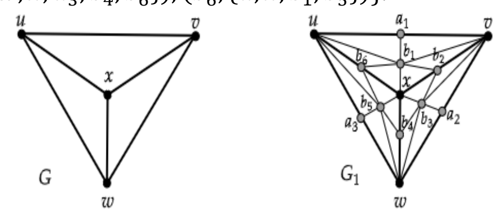

Figure 9: A wheel graph G and its first barycentric refinement graph G_1.

4.3 Trees

In a tree graph T of order n, there are exactly $n-1$ edges and so to find first barycentric refinement $S(T)$ of T, we need to perform exactly $n-1$ vertex-insertion operations.

Proposition 5. Let (F_T, V) be the SSR of a tree graph $T = (V, E)$. Then, there are at least two vertices, say u and v such that their F_G-images contains exactly one element, that is $|F_G(u)| = |F_G(v)| = 1$.

Lemma 8. Let $G = (V, E)$ be a tree graph such that $|V| = n$ and $G_1 = (V_1, E_1)$ be its first barycentric refinement. Then

1. $|F_{G_1}(v)| = 1$ if v is a leaf vertex in G_1 and $|F_{G_1}(v)| \geq 2$ if v is not a leaf in G_1.
2. $|\mathcal{F}_{G_1}| = 2n - 1$.
3. $|E_{G_1}| = 2(n-1)$.

Proof. (1) Since G is acyclic, then clearly $S(G) = G_1$ is also acyclic. This means that G_1 is also a tree. Thus, the required follows directly.
(2) Since G is acyclic, so $S(G) = G_1$. But $|E| = n - 1$, so to reach G_1, we need to perform $n-1$ edge-subdivision operations. Hence, G_1 has exactly $n + (n-1) = 2n - 1)$ vertices.

Example 9. Consider the tree graph $G = (V, E)$ in Figure 10. The SSR of G is
$(F_G, V) = \{(x, \{\mathrm{u}, v, w\}), (u, \{x, r\}), (v, \{x, s\}), (w, \{x, t\}),$
$(r, \{u\}), (s, \{v\}), (t, \{w\})\}$.
$(F_{G_1}, V_1) = \{(x, \{a_1, a_2, a_3\}), (u, \{a_1, b_1\}), (v, \{a_2, b_2\}),$
$(w, \{a_3, b_3\}), (a_1, \{u, x\}), (a_2, \{v, x\}), (a_3, \{w, x\}),$
$(b_1, \{\mathrm{u}, \mathrm{r}\}), (b_2, \{\mathrm{v}, \mathrm{s}\}), (b_3, \{\mathrm{w}, \mathrm{t}\}), (r, \{b_1\}), (s, \{b_2\}), (t, \{b_3\})\}$.

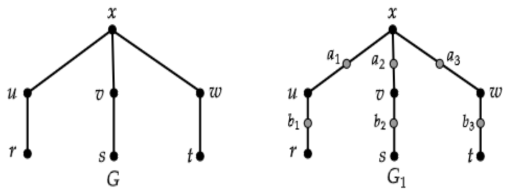

Figure 10: A tree graph G and its first barycentric refinement graph G_1.

4.4 Star Graphs

Recall that a star graph S_n is a tree graph with n vertices such that one vertex, say $s \in V$, is of degree $n-1$ and each other vertex other than s is of degree 1. The vertex s is called the hup vertex of S_n.

Lemma 9. The first barycentric refinement G_1 of a star graph G with n vertices can be obtained from G by performing $n-1$ vertex-inserting operations on G.

Proof. Since G has no cycle, so there is no triangle. Thus, only subdivision operations are needed to obtain G_1. Thus, $G_1 = S(G)$. But, G has $n-1$ edges. Hence, $S(G)$ can be obtained by performing $n-1$ vertex-inserting operations.

Proposition 6. Let (F_G, V) be the *SSR* of a star graph $G = (V, E)$ with n vertices and let s be its hup vertex. Assume that (F_{G_1}, V_1) be the *SSR* of $G_1 = (V_1, E_1)$ (The first barycentric refinement of G). Then the following are held:

1. $\left|F_{G_1}(s)\right| = |F_G(s)| = n-1$ where s is the hup of G, and $\left|F_{G_1}(v)\right| = 2$ for all subdivided vertices of G and $\left|F_{G_1}(v)\right| = 1$ for $v \in V - \{s\}$.
2. $\left|\mathcal{F}_{G_1}\right| = 2n-1$.
3. $|E_{G_1}| = 2(n-1)$.

Proof. (1) Since G is a also a tree, so $S(G) = G_1$. This means the hup vertex (2) and (3) follow from being G is also a tree, hence by Lemma 8(2) and (3), the required is followed directly.

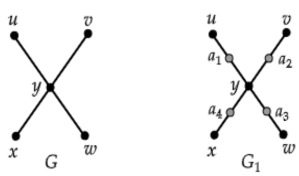

Figure 11: A star graph G and its first barycentric graph G_1.

Example 10. Consider the star graph $G = (V, E)$ which is in Figure 11. The *SSR* of G is
$(F_G, V) = \{(u, \{y\}), (v, \{y\}), (w, \{y\}), (x, \{y\}), (y, \{\mathrm{u}, \mathrm{v}, \mathrm{w}, \mathrm{x}\})\}$.
$\left(F_{G_1}, V_1\right) = \{(u, \{a_1\}, (\mathrm{v}, \{a_2\}, (w, \{a_3\}), (\mathrm{x}, \{a_4\}),$
$(\mathrm{y}, \{a_1, a_2, a_3, a_4\}), (\, a_1, \{u, y\}), (\, a_2, \{v, y\}),$
$c(\, a_3, \{w, y\}), (\, a_4, \{x, y\})\}$.

4.5 Bipartite Graphs

A graph $G = (V, E)$ is called bipartite graph if $V = M \cup N$, $M \cap N = \emptyset$, $|M| = m$, $|N| = n$ and for each edge $e_{uv} \in E$ implies that $u \in M$ and $v \in N$. A complete bipartite graph $K_{m,n} = (V, E)$ is a bipartite graph such that for each $u \in M$ and each $v \in M$, then $e_{uv} \in E$.

Lemma 10. [10] There is no odd cycle in $K_{m,n}$.
Notice that the first barycentric refinement of a complete bipartite graph $K_{m,n}$ can be obtained by only find the subdivision of $K_{m,n}$.

Proposition 7. Let (F_G, V) be the *SSR* of a complete bipartite graph $K_{m,n}$=$G = (V, E)$ with where $V = M \cup N$. Assume that (F_{G_1}, V_1) be the *SSR* of $G_1 = (V_1, E_1)$ (The first barycentric refinement of G). Then the following are held:

1. $|F_{G_1}(u)| = n$ for each $u \in M$ and $|F_{G_1}(v)| = m$ for each $v \in N$.
2. $\left|F_{G_1}(y)\right| = 2$ for each subdivided vertex y in G_1.
3. $\left|\mathcal{F}_{G_1}\right| = m + n + mn$.
4. $|E_{G_1}| = 2mn$.

Proof. (1) Let $u \in M$. By Lemma 10, G has no 3-cycle. Hence $S(G) = G_1$. But $|F_G(u)| = n$, this means that each adjacent edge for u will be subdivided. Thus, the neighbourhood set of u in G_1 is of size equals to the neighbourhood set of u in G. Therefore, $|F_{G_1}(u)| = n$.
(2) This follows from being $S(G) = G_1$.
(3) Since G has no 3-cycle, so $S(G) = G_1$. But G has nm edges, so we need exactly nm edge-subdivision operations to obtain G_1. Hence, nm new vertices will be introduced eventually. Consequently, $\left|\mathcal{F}_{G_1}\right| = m + n + mn$.
(4) Since $S(G) = G_1$ and G has nm edges, so we need exactly nm edge-subdivision operations to obtain G_1. Therefore, the number of the edges of G_1 is as twice as of G.

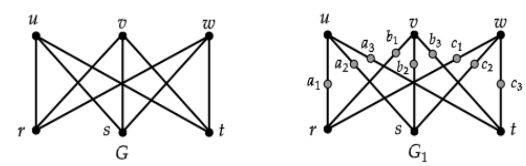

Figure 12: The complete bipartite graph $G = K_{3,3}$ and its first barycentric refinement G_1.

Example 11. Consider the complete graph $G = (V, E)$ which is $K_{3,3}$ in Figure 12. The *SSR* of G is
$(F_G, V) = \{(u, \{\mathrm{r}, s, t\}), (v, \{r, s, t\}), (w, \{r, s, t\}), (r, \{u.v.w\}),$
$(s, \{u.v.w\}), (\mathrm{t}, \{u.v.w\}\}$.
$\left(F_{G_1}, V_1\right) = \{(u, \{a_1, a_2, a_3\}), (v, \{b_1, b_2, b_3\}), (w, \{c_1, c_2, c_3\}),$
$(r, \{a_1, b_1, c_1\}), (s, \{a_2, b_2, c_2\}), (\mathrm{t}, \{a_3, b_3, c_3\}), (a_1, \{\mathrm{u}, \mathrm{r}\}),$
$(a_2, \{\mathrm{u}, \mathrm{s}\}), (a_3, \{\mathrm{u}, \mathrm{t}\}), (b_1, \{\mathrm{v}, \mathrm{r}\}), (b_2, \{\mathrm{v}, \mathrm{s}\}), (b_3, \{\mathrm{v}, \mathrm{t}\}),$
$(c_1, \{\mathrm{w}, \mathrm{r}\}), (c_2, \{\mathrm{w}, \mathrm{s}\}), (c_3, \{\mathrm{w}, \mathrm{t}\})\}$.

Conclusion

Within this work, we investigated the first barycentric refinements of simple graphs in term of soft set representations. Specifically, we adopted representing graphs using soft sets to represent the first barycentric of simple graphs that contains cliques of sizes less than four. Some interested properties and related results of such representations have been found. We also studied representing the first barycentric refinements of some known simple graphs using soft set.

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