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Original Article

Reflection Spaces Related to Hyperbolic Planes

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ABSTRACT

This research investigates the algebraic and geometric properties of reflection spaces within the framework of general hyperbolic planes. While the theory of reflection spaces, as established by Karzel and Taherian, has been applied extensively to elliptic planes, its application to hyperbolic geometry has remained largely unexplored. We provide a comprehensive analysis of the involution-based structures generated by point reflections, line reflections, and their collective union within the motion group of a hyperbolic plane. Our findings reveal a tiered structural hierarchy in which point reflections alone constitute a polar-free reflection space that maintains an isomorphism with the plane's original incidence structure. In contrast, the set of line reflections yields a reducible, polar reflection space that recovers the dual geometric structure. Most significantly, we demonstrate that the union of point and line reflections results in a maximal reducible polar reflection space. Under the condition that the hyperbolic plane is complete, we extend the incidence structure of this maximal space and prove its embeddability into a projective plane, where the boundary set of ends forms a geometric oval.

Keywords: Reflection Space, Hyperbolic Geometry, Point Reflection, Line Reflection, Group Theory.

1 Introduction

Over the past two centuries, reflections in points and lines have been foundational in the development of geometric methods and the proof of fundamental theorems. An early milestone is due to Schur, who in 1899 derived a reflection-based proof of Pappus' theorem using point reflections [1]. Shortly thereafter, Hilbert introduced line reflections in his construction of hyperbolic planes (1903), while Hessenberg applied related ideas to the opposite-pairs theorem.

Hjelmslev made extensive use of reflection calculations in his foundation of absolute planes [2]. Subsequently, Podehl and Reidmeister successfully used reflections in the foundation of elliptic planes by constructing an abstract kinematic space. These developments led Thomsen in 1933 to give an axiomatization of the Euclidean plane based on reflection theory [3].

Group-theoretic representations of non-Euclidean planes emerged a decade later through the work of various mathematicians. A reflection-theoretic axiom system that establishes elliptic and also non-elliptic absolute planes was introduced by A. Schmidt [4]. In Schmidt's system, P and $\boldsymbol{G}$ are not necessarily disjoint; and he formulated several axioms, including the existence of joining lines, the existence of perpendiculars, uniqueness conditions, and three reflection theorems for point bundles and plumb bundles.

Bachmann showed in 1951 that Schmidt's system was not independent and reduced it [5]. In 1952, Bachmann published his comprehensive book presenting a collective representation of the theory of absolute planes based on reflection theory.

The next significant development was influenced by Sperner's axiom system in 1954 [6]. Sperner aimed to determine under what minimal conditions one could prove Desargues' theorem using group-theoretical tools. He started with a group G and an involutory generator set $\boldsymbol{E}$ claiming essentially only the general three-reflection theorem. Sperner's work established conditions under which classical geometric theorems could be proven in this abstract setting.

It remained an open question for many years whether Sperner's geometry could be embedded within projective geometry or if its motion group could be represented as a subgroup of an orthogonal group. Kannenberg proved in his dissertation [7] that "if Sperner's geometry can be embedded in a projective plane with characteristic $\neq 2$, then the corresponding skew-field K is a

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field and there exists a quadratic form $Q: K^3 \to K$ such that line reflections can be represented as usual". Similar results were established by Karzel for the case of characteristic $\neq 2$ in 1955 [8, 9].

Building on this extensive historical background, Karzel and Taherian [10, 11] established the concept of a "reflection space". In addition, they investigated the characteristics of reflection geometries and their associated group spaces in [12].

We now turn to reflection spaces and briefly summarize their axiomatic formulation. Let Π be a group with identity element 1, $\boldsymbol{J}$ be the subset of all its involutions, and P be a non-empty subset of $\boldsymbol{J}$. Similar to [13], Karzel and Taherian defined the following subsets of P in [10]:

"For $m, n \in P$ with $m \neq n$, let

$$\widetilde{m,n} := \begin{cases} \{x \in P \mid m \cdot n \cdot x \in \boldsymbol{J}\} & \text{if } m \cdot n \neq n \cdot m \\ \{x \in P \mid m \cdot n \cdot x \in \boldsymbol{J} \wedge m \cdot x \neq x \cdot m\} \cup \{n\} & \text{if } m \cdot n = n \cdot m \end{cases}$$

Furthermore, we define $\Pi(m) := \{x \in P \setminus \{m\} \mid x \cdot m = m \cdot x\}$. We call $\Pi(m)$ *polar* of m."

We call the elements of $\boldsymbol{L} := \{\widetilde{m,n} \mid m, n \in \binom{P}{2}\}$ *lines.* Now, (P, Π) is said to be a "*reflection space"* whenever the axioms (**S1**)–(**S3**) listed below are satisfied [10]:

"(**S1)** The set P generates the group Π."

"(**S2**) (Three-Reflection Axiom) For any $L \in \boldsymbol{L}$ and all $r, s, t \in L$, we have $r \cdot s \cdot t \in P$."

"(**S3**) There are $r, s, t \in P$ such that $r \cdot s \cdot t \notin \boldsymbol{J}$."

Moreover, we consider the following axiom:

"(**S4**) (Polar Axiom) For all $m, n, p, q \in P$ with $n \neq p$ if $m \cdot n, m \cdot p, n \cdot p \cdot q \in \boldsymbol{J}$ then $m \cdot q \in \boldsymbol{J}$ [10]."

Also, in [10] "a reflection space (P, Π) is called *polar* if there exists an element $p \in P$ with $\Pi(p) \neq \emptyset$, and *polar-free* otherwise". In addition, it is called "*reducible"*, if

$$\forall x, y, z, w \in P\ \exists m, n \in P: x \cdot y \cdot z \cdot w = m \cdot n.$$

A reflection space that only satisfies the axioms **(S1)** and **(S2)** is said to be a *degenerated* reflection space [10].

In [13], reflection spaces satisfying some further conditions are called *reflection groups*.

In [14], Karzel and Taherian employed point reflections in an elliptic space (using Sörensen approach [15,16]) to obtain a reflection space (P, Π) where its associated incidence structure $(P, \boldsymbol{L})$ forms a projective space. However, reflection spaces related to a hyperbolic plane have not yet been studied.

Reflection spaces for hyperbolic planes unify point and line reflections within a single algebraic framework, revealing structural properties of the motion group. The embedding into projective planes connects hyperbolic and projective geometry, providing insights into compactifications and boundary structures relevant to symmetric spaces and hyperbolic manifolds.

The notion of quasi-end, a pencil of mutually quasi-parallel lines, was introduced and used to classify general absolute planes by Karzel, Pianta, Rostamzadeh, and Taherian [17], where it is shown that for hyperbolic planes every line belongs to exactly two quasi-ends. Criteria for quasi-parallelism in absolute planes were further developed in [18].

Recent developments have continued to advance the foundations of hyperbolic and absolute geometry from complementary directions. Augat and Hähl [19] presented a new axiom system for absolute plane geometry in terms of collinearity and congruence, studying in particular the system of line reflections and motions, and showing that the hyperbolic axiom recovers the classical algebraic models. Bamberg and Penttila [20] gave simpler first-order foundations for the hyperbolic plane as a pure incidence geometry, while Pambuccian, Struve, and Struve [21] employed reflection-geometric proofs, relying centrally on the Three-Reflection Theorem, to analyze axiomatic frameworks in metric planes. These works confirm that reflection theory remains a central and active tool in the foundations of non-Euclidean geometry, and motivate the present study of reflection spaces arising from the motion group of a hyperbolic plane.

In the present work, we consider a general hyperbolic plane in the sense of Karzel [13] and study its associated reflection spaces using the point and the line reflections. By "general" we mean that there is no assumption made on continuity or the Archimedean Axiom. In Section 4, we additionally assume that the hyperbolic plane is complete in the sense that any two distinct quasi-ends can be joined by a unique line. By Bachmann [5] (§16,3, p. 211), this is equivalent to the Axiom of Continuity (Dedekind's Axiom). Let $\mathfrak{M}$ be the motion group of the hyperbolic plane and $\boldsymbol{J}$ be the set of all its involution elements, $\widetilde{\boldsymbol{E}}$ represents point reflections, and $\widetilde{\mathfrak{G}}$ denotes the set of all line reflections, then $\boldsymbol{J} = \widetilde{\boldsymbol{E}} \cup \widetilde{\mathfrak{G}}$ (cf. [13], (17.11)). Therefore, we can establish the pair (P, Π) with $\Pi := \langle P \rangle, P := \widetilde{\boldsymbol{E}}, P := \widetilde{\mathfrak{G}}$, or $P := \boldsymbol{J} = \widetilde{\boldsymbol{E}} \cup \widetilde{\mathfrak{G}}$, and ask these two questions: Which one(s) of these pairs (P, Π) is a reflection space and what are their properties. From now and on, throughout the paper, we identify points $\boldsymbol{E}$ and lines $\mathfrak{G}$ with their corresponding reflections in the motion group; hence products such as $a \cdot b$ denote composition of the associated involutions.

Setting $P = \boldsymbol{E}$, we demonstrate in Theorem 3.1 that (P, Π) constitutes a polar-free reflection space. Furthermore, we establish that its incidence space is identical to the incidence structure $(\boldsymbol{E}, \mathfrak{G})$ of the hyperbolic plane. For $P = \mathfrak{G}$, we have

$\Pi = \mathfrak{M}$ and the pair $(\mathfrak{G}, \mathfrak{M})$ is a polar and reducible reflection space (cf., Theorem 3.2). Under these conditions, the incidence geometry is isomorphically equivalent to the dual hyperbolic plane. Finally, we examine the maximal case where $P = \boldsymbol{J} = \boldsymbol{E} \cup \mathfrak{G}$. As shown in Theorem 3.6, (P, Π) is both a polar and a reducible reflection space. Here, the points of the associated incidence space $(P, \boldsymbol{L})$ consist of the totality of points and lines from the hyperbolic plane, hence $P = \boldsymbol{E} \cup \mathfrak{G}$.

We use lowercase letters to represent points and uppercase letters to denote lines. Two orthogonal lines S and T are shown by $K \perp T$, the line perpendicular to W from a point p is represented by $(p \perp W)$, and $W^{\perp} = \{X \in \mathfrak{G} | X \perp W\}$ stands for the set of all lines perpendicular to W. We recall definition of two quasi-parallel lines from [17]: "two distinct lines C and D are called *quasi-parallel* and denoted by $C \ddagger D$, if $C \cap D = \emptyset$ and $C^{\perp} \cap D^{\perp} = \emptyset$, hence they neither intersect nor have a common perpendicular". In [18], criteria for quasi-parallel lines are studied. For the lines A and B, the set $\overbrace{A, B} := \{X \in \mathfrak{G} \,| A \cdot B \cdot X \in \boldsymbol{J}\}$ is said to be a *pencil*. In [17], "a pencil is called a *quasi-end* if its lines are mutually quasi-parallel".

The lines of the corresponding incidence space of the maximal reflection space $(\boldsymbol{E} \cup \mathfrak{G}, \Pi)$ are three types:

1. $\boldsymbol{L}_{\mathrm{p}} = \mathfrak{G}(p)$: the pencil of lines incident with a specific point.
2. $\boldsymbol{L}_{\mathrm{e}} = A \cup A^{\perp}$: The set consists of points of a line combined with the set of all lines orthogonal to it.
3. $\boldsymbol{L}_{\boldsymbol{g}}$: the set of all quasi−ends.

In the hyperbolic plane, the set $\mathcal{E}$ of quasi-ends is not empty. We prove that the incidence space $(P, \boldsymbol{L})$ of the maximal reflection space cannot be a projective plane (cf., Subsection 4.2), however one can extend it to a projective plane by extending the set $\boldsymbol{J}$ of points to $\Phi := \mathbf{J} \cup \mathcal{E}$, and the lines of $\boldsymbol{L}$ to $\bar{\boldsymbol{L}}$. In a complete hyperbolic plane $\mathcal{A}$ is, i.e., any two quasi-ends can be joined by a line, we show that $(\Phi, \bar{\boldsymbol{L}})$ is a projective plane (cf., Theorem 4.3). Moreover, $\mathcal{E}$ is an oval. (Theorem 4.3).

2 Basic notions and known results

2.1 Incidence spaces

We recall from [10,13] that "a pair $(P, \boldsymbol{L})$ consisting of a set P -the set of *points*- and a set $\boldsymbol{L}$ of subsets of the set P -the set of *lines*- is called an *incidence space* if any two distinct points lie on a unique line $L \in \boldsymbol{L}$ and if $\forall L \in \boldsymbol{L}, |L| \geq 2$." Also, "a subset $F \subseteq P$ is called *subspace* if for all $x, y \in F$ with $x \neq y, \overline{x, y} \subseteq F$, in which $\overline{x, y}$ is the unique line passing through the points x and y". The collection $\mathfrak{S}$ of all subspaces within an incidence space is closed under intersection; specifically, the intersection of any family of subspaces is itself a subspace., and therefore, for $T \subseteq P$ we can form the *incidence closure*

$$\overline{\overline{T}} := \bigcap\{S \in \mathfrak{S} | T \subseteq S\}$$

which is a subspace. In [13], "a set consisting of three non-collinear points a, b, and c, the incidence closure of $\{a, b, c\}$, denote by $F = \overline{\overline{\{a, b, c\}}}$ is called a *plane*".

For a subspace $s \in \mathfrak{S}$ let $\boldsymbol{L}(s) := \{L \in \boldsymbol{L} \mid L \subseteq s\}$. Therefore, $(s, \boldsymbol{L}(s))$ is an incidence space. "A line $A \in \boldsymbol{L}$ is called *projective* if for any plane F with $A \subseteq F$ and $L \in \boldsymbol{L}(F)$, we have $A \cap L \neq \emptyset$" (cf. [13]).

Definition 2.1 An incidence space $(P, \boldsymbol{L})$ is called a projective plane if the following axioms hold:

P1. Any two distinct points determine a unique line.

P2. Any two distinct lines meet in a point.

P3. There exist at least four points of which no three are collinear.

Definition 2.2. In a projective plane a set $\mathcal{E}$ of points is called an oval, if:

1. Any line L meets $\mathcal{E}$ in at most two points, and

2. For any point $\varepsilon \in \mathcal{E}$ there exists exactly one tangent line L through ε, i.e., $L \cap \mathcal{E} = \varepsilon$.

2.2 Some Properties of Reflection Spaces

We recall the following statements from [10]:

Theorem 2.1 *For a reflection space* (P, Π)*:*

1. $(P, \boldsymbol{L})$ *forms an incidence space.*
2. $\forall T \in \boldsymbol{L},\ T^3 = \{t_1 \cdot t_2 \cdot t_3 \,| t_1, t_2, t_3 \in T\} = T$, and T^2 forms a commutative group.
3. $\forall m, s, n \in P: m \cdot s \cdot n \in \boldsymbol{J} \iff m \cdot s \cdot n \in P$.
4. $\forall \sigma \in \Pi, \forall r, d \in P, r \neq d, \forall R \in \mathbf{L}$:

$$\sigma \cdot r \cdot \sigma^{-1} \in \overbrace{\sigma \cdot r \cdot d \cdot \sigma^{-1}} = \sigma \cdot \overbrace{r \cdot d} \cdot \sigma^{-1},\ \sigma \cdot R^2 \cdot \sigma^{-1} = (\sigma \cdot R \cdot \sigma^{-1})^2.$$

2.3 Absolute Planes

As discussed in [13], hyperbolic planes belong to the class of absolute planes. In this section, we recall the foundational properties of an absolute plane $\mathcal{A} = (\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$, following the definitions in [13], where $\boldsymbol{E}$ represents the set of points, $\mathfrak{G}$ the set of lines, α the order structure, and $\equiv$ the congruence relation.

We adopt the convention of using lowercase letters for points and uppercase letters for lines. The orthogonality of two lines A and T is denoted by $A \perp T$, while $(t \perp T)$ represents the unique line through point t perpendicular to T. We define $A^{\perp} = \{X \in \mathfrak{G} | X \perp A\}$ as the set of lines perpendicular to A, and $\mathfrak{G}(m) := \{X \in \mathfrak{G} \mid m \in X\}$ as the pencil of lines incident with point m.

We recall from [13] that "a *motion* of the absolute plane $\mathcal{A}$ is a permutation μ of the point set $\boldsymbol{E}$ that preserves the line structure $\mathfrak{G}$, the order α, and the congruence $\equiv$ of $\mathcal{A}$, and moreover, if $a, b \in \boldsymbol{E}$ then $(\mu(a), \mu(b)) \equiv (a, b)$." The set of all motions forms the *motion group* $\mathfrak{M}$. Let $\boldsymbol{J} := \{\mu \in \mathfrak{M} \mid \mu^{2} = id \neq \mu\}$ denotes the set of all involutions in $\mathfrak{M}$. Each point $p \in \boldsymbol{E}$ corresponds to a unique point reflection $\tilde{p} \in \boldsymbol{J}$ which fixes exactly the point p, and each line $G \in \mathfrak{G}$ corresponds to a unique line reflection $\tilde{G} \in \boldsymbol{J}$ which fixes exactly the points of the line G. By identifying points and lines with their respective reflections, $\boldsymbol{E}$ and $\mathfrak{G}$ become subsets of $\boldsymbol{J}$, allowing geometric relations to be expressed as group equations.

Theorem 2.2 Let $\mathcal{A} = (\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$ be an absolute plane, $n, b \in \boldsymbol{E}$, and $W, N \in \mathfrak{G}$. The following algebraic characterizations hold:

1. **Incidence**: $b \in W \Leftrightarrow b \cdot W \in \boldsymbol{J} \Leftrightarrow b \cdot W \in \mathfrak{G}$. *Therefore*, $\mathfrak{G}(b) = \{X \in \mathfrak{G} \mid b \cdot X \in \boldsymbol{J}\}$ (cf., 11.2 of [13]).
2. **Orthogonality**: $W \perp N \Leftrightarrow W \cdot N \in \boldsymbol{J} \Leftrightarrow W \cdot N \in \boldsymbol{E}$ (cf., 11.1 of [13]).
3. **Distinctness**: If $b \neq n$, then $b \cdot n \notin \boldsymbol{J}$ (cf., 11.1 of [13]).
4. **Reflections**:
   i. Point reflection in b: $\tilde{b}: \boldsymbol{E} \to \boldsymbol{E}; x \mapsto b \cdot x \cdot b$ (cf., 11.2 of [13]).
   ii. Line reflection in W: $\widetilde{W} : \boldsymbol{E} \to \boldsymbol{E};\ x \mapsto W \cdot x \cdot W$ (cf., 11.2 of [13]).

In [13], "for two distinct lines $R, N \in \mathfrak{G}$ the set $\overbrace{R, N} := \{X \in \mathfrak{G} \,| R \cdot N \cdot X \in \boldsymbol{J}\}$ of lines is called a *pencil*".

We recall from [17] that "two distinct lines $A, B \in \mathfrak{G}$ are said to be *quasi-parallel* if $A \cap B = \emptyset$ and $A^{\perp} \cap B^{\perp} = \emptyset$ which we denote by $A \ddagger B$". Also, "a pencil $\mathfrak{g} \subseteq \mathfrak{G}$ is called *quasi-end* if for any distinct lines $A, B \in \mathfrak{g}$ we have $A \ddagger B$." Let $\mathcal{E}$ represents the set of all quasi-ends of the absolute plane $\mathcal{A}$, and set $\mathfrak{G}_{\ddagger} := \{X \in \mathfrak{G} \mid \exists Y \in \mathfrak{G} : Y \ddagger X\}$ to be the set of lines which have a quasi-parallel line. For a line A, define $A_{\mathcal{E}} := \{\varepsilon \in \mathcal{E} \mid A \in \varepsilon\}$ as the set of all quasi-ends containing the line A, and $\omega_{\mathcal{A}}(A) := |A_{\mathcal{E}}|$ be the cardinal-number of $A_{\mathcal{E}}$. In [17], "an absolute plane $\mathcal{A}$ is called *complete* if any two quasi-ends can be joined by a line, and *half-elliptic* if $\mathfrak{G}_{\ddagger} = \emptyset$, i.e., if there are no quasi-parallel lines". In addition to the foundational properties discussed above, we utilize the following results established in [17]:

If $A \in \mathfrak{G}_{\ddagger}$ then A is contained in at least two quasi-ends, hence $\omega_{\mathcal{A}}(A) \geq 2$.

By theorems (16.11) and (16.12) of [13], any two distinct points $x, y \in \boldsymbol{E}$ have a unique midpoint m, and exactly one midline M, and so, $C := m \cdot M = \overline{x, y}$. Therefore, the motion group $\mathfrak{M}$ acts transitively on $\boldsymbol{E}$ and transitively on $\mathfrak{G}$ that implies $\mathfrak{G}_{\ddagger} = \mathfrak{G}$ if $\mathfrak{G}_{\ddagger} \neq \emptyset$. The number of quasi-ends incident with a line remains constant across all lines, serving as a plane invariant for the classification of absolute planes as follows:

- $\omega_{\mathcal{A}} = 0$. The absolute plane in [17] is called *half-elliptic* and,
- $\omega_{\mathcal{A}} \geq 2$. It is proved in [17] the cardinality cannot be 1 and in the singular case $\omega_{\mathcal{A}}$ must be infinite. The case $\omega_{\mathcal{A}} = 2$ is called *quasi-hyperbolic* absolute plane. In Cor. 5.1 of [17], it is proved that $\omega_{\mathcal{A}} = 2$.

In [13], absolute planes are classified into two classes: "Singular absolute planes in which for all $r, s, q \in \boldsymbol{E}, r \cdot s \cdot q \in \boldsymbol{E}$, and ordinary absolute planes in which there are points $r, s, q \in \boldsymbol{E}$ with $r \cdot s \cdot q \notin \boldsymbol{E}$". We need the following results from [13] for later theorems and proofs:

Theorem 2.3 *Let* $\mathcal{A} = (\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$ *be an absolute plane. Then:*

1. $\mathfrak{M} = <\mathfrak{G}> = \mathfrak{G}^2 \,\dot{\cup}\, \mathfrak{G}^3$, $\boldsymbol{E} \subseteq \mathfrak{G}^2$ and $\mathfrak{M}^{+} := \{X \cdot F \mid X, F \in \mathfrak{G}\}$ is a subgroup of $\mathfrak{M}$ of index 2 which is called "the group of proper motions" (cf., (17.6), (17.8), (17.9) of [13]).
2. There are lines $R, S, T \in \mathfrak{G}$ *with* $R \cdot S \cdot T \notin \boldsymbol{J}$ (cf. [13], (13.4)).
3. For two distinct points x *and* m, $\overline{x, m} \subseteq \{z \in \boldsymbol{E} \mid x \cdot m \cdot z \in \boldsymbol{J}\}$, *and* $x \cdot m \notin \boldsymbol{J}$. *If* $\mathcal{A}$ *is singular then* $\boldsymbol{E} = \{z \in \boldsymbol{E} | x \cdot m \cdot z \in \boldsymbol{J}\}$, *and if* $\mathcal{A}$ *is ordinary then* $\overline{x, m} = \{z \in \boldsymbol{E} \mid x \cdot m \cdot z \in \boldsymbol{J}\}$.
4. $\boldsymbol{J} = \boldsymbol{E} \,\dot{\cup}\, \mathfrak{G}$ (cf. [13], (17.11)).
5. (*Reduction Theorem*) *For any* $M, N, P, D \in \mathfrak{G}$ *there exist* $Q, T \in \mathfrak{G}$ *such that* $M \cdot N \cdot P \cdot D = Q \cdot T$ (cf., [13], (19.5)).
6. *For* $p \in \boldsymbol{E}$ *there exist* $A, B \in \mathfrak{G}$ *such that* $p = A \cdot B = B \cdot A$ (then $A \perp B$), i.e., $A \cdot B \in \boldsymbol{J}$. *Also, for any* $A, B, C \in \mathfrak{G}$ with $A \cdot B \cdot C \neq 1$ *then* $Fix(A \cdot B) = \{A \cap B\}$(cf. [13], (17.7)).
7. *If* $G, H, K \in \mathfrak{G}$ *and* $G, H \perp K$ *with* $G \neq H$, *then* $G \cap H = \emptyset$ (cf., 16.10, [13]).

Theorem 2.4 (Three Reflection Theorems) *Let* $p \in \boldsymbol{E}, A, B, X, Y, Z \in \mathfrak{G}$ *with* $A \neq B$. *Then:*

1. *(General Three Reflection Theorem) If* $A \cdot B \cdot X, A \cdot B \cdot Y, A \cdot B \cdot Z \in \boldsymbol{J}$ *then* $X \cdot Y \cdot Z \in \mathfrak{G}$ (cf., [13], (18.8)).
2. *If* $p \in A \cap B$ *then:* $A \cdot B \cdot X \in \boldsymbol{J} \Leftrightarrow p \in X \Leftrightarrow \exists M \in \mathfrak{G}$ *with* $M = A \cdot B \cdot X$ *and* $p \in M$ (cf., [13], (17.13) and (17.14)).
3. *If* $A, B \perp Y$ *then:* $A \cdot B \cdot X \in \boldsymbol{J} \Leftrightarrow X \perp Y \Leftrightarrow \exists M \in \mathfrak{G}$ *with* $M = A \cdot B \cdot X$ *and* $M \perp Y$ (cf., [13], (17.13) and (17.14)).
4. *If* $A \ddagger B$ *then*: $A \cdot B \cdot X \in \boldsymbol{J} \Leftrightarrow \exists M \in \mathfrak{G}$ *with* $M = A \cdot B \cdot X$ *and* $\mathrm{M} \in \overbrace{A, B}$. *Besides,* $\overbrace{A, B}$ *is a quasi-end* (cf., [13], (19.1.2), (18.7)).

Lemma 2.1 *For any three distinct lines $R, N, P \in \mathfrak{G}$ there are $p \in \boldsymbol{E}$ and $L \in \mathfrak{G}$ such that $R \cdot N \cdot P = L \cdot p$, or $R \cdot N \cdot P = p \cdot L$.*

Proof. We examine four scenarios based on line configuration:

(i) If $P \in \widetilde{\widetilde{R, N}}$ then by Theorem 2.4(1) there exists an $S \in \mathfrak{G}$ such that $R \cdot N \cdot P = S$. Let $p \in S$ and $L := (p \perp S)$, then by Theorem 2.3(6), $R \cdot N \cdot P = L \cdot L \cdot R \cdot N \cdot P = L \cdot L \cdot S = L \cdot p$.

(ii) If $R \cap N =: d$ and $D := (d \perp P)$, then by Theorem 2.4(2), there exists $L \in \mathfrak{G}$ with $d \in L$ and $R \cdot N \cdot D = L$. Therefore, $R \cdot N \cdot C = (R \cdot N \cdot D) \cdot (D \cdot C) = L \cdot p$ for $p := P \cap D$.

(iii) If $R, N \perp S$ for $S \in \mathfrak{G}$, let $d \in P$ and $D := (d \perp S)$, then, by Theorem 2.4(3) there exists an $E \in \mathfrak{G}$ with $E \perp S$ satisfying $R \cdot N \cdot D = E$, hence $R \cdot N \cdot P = R \cdot N \cdot D \cdot D \cdot P = E \cdot D \cdot P$. Let now $W := (d \perp E)$ and $L \in \mathfrak{G}$ with $W \cdot D \cdot P = L$, so $E \cdot D \cdot P = (E \cdot W) \cdot (W \cdot D \cdot P) = p \cdot L$ for $p := E \cap W$.

(iv) If $R \ddagger N$, let $d \in P$ and D be the unique line through d belonging to $\widetilde{\widetilde{R, N}}$ (cf., [13], (18.6)). By part 4 of Theorem 2.4, there exists a line $S \in \mathfrak{G}$ with $S \ddagger R, N$ with $R \cdot N \cdot D = S$. So, we can decompose $R \cdot N \cdot P = (R \cdot N \cdot D) \cdot (D \cdot P) = S \cdot D \cdot P$. Now, since $D \cap P = d$, proceeding as in case (ii), we can simplify the product $S \cdot D \cdot P$ to find a point p and a line L such that $R \cdot N \cdot P = S \cdot D \cdot P = p \cdot L$. □

Theorem 2.5 *For points $n, r, p \in \boldsymbol{E}$,* and lines $N, R, P \in \mathfrak{G}$ *with $r \neq n$ and $R \neq N$ we have:*

1. $\exists p \in \boldsymbol{E}$ with $R \cdot N \cdot p \in \boldsymbol{J} \Leftrightarrow R^{\perp} \cap N^{\perp} \neq \emptyset$.
2. $r \cdot n \cdot P \in \boldsymbol{J} \Leftrightarrow r \cdot n \cdot P \in \mathfrak{G} \Leftrightarrow P \perp \overline{r, n}$.
3. If $R, N \perp P$ and $p \in P$ then $R \cdot N \cdot p \in P$ and $p \cdot R \cdot N \in P$.

Proof. 1. Let $R \cdot N \cdot p \in \boldsymbol{J}$. From $\boldsymbol{J} = \boldsymbol{E} \cup \mathfrak{G}$ and $R \cdot N \cdot p \in \mathfrak{G}^4$, a point $d \in \boldsymbol{E}$ other than p exists with $R \cdot N \cdot p = d$. Let $L := \overline{p, d}$, $Q := (p \perp L)$, $C := (d \perp L)$. Then $R \cdot N = d \cdot p = C \cdot P$. Therefore, $N, R \perp L$, i.e., $L \in R^{\perp} \cap N^{\perp}$.

Conversely, suppose $L \in R^{\perp} \cap N^{\perp}$. For $c \in L$ and $Q = (p \perp L)$, Theorem 2.4 guarantees that a line $C \in L^{\perp}$ with $R \cdot N \cdot Q = C$ exists, hence $R \cdot N \cdot p = R \cdot N \cdot (Q \cdot L) = C \cdot L \in \boldsymbol{J}$.

2. From $r \cdot n \cdot P \in \mathfrak{G}^5$ we have: $r \cdot n \cdot P \in \boldsymbol{J} \Leftrightarrow r \cdot n \cdot P \in \mathfrak{G}$. Let $L := \overline{r, n}$, $R := (r \perp L), N := (n \perp L)$. Since R, N, and P are perpendicular to L, $r \cdot n \cdot P = (R \cdot L)(L \cdot N) \cdot P = R \cdot N \cdot P \in \boldsymbol{J}$ implying $P \perp L$ (by Three Reflection Theorem).

3. We express the product of the line reflections as $R \cdot N = (R \cdot P) \cdot (P \cdot N)$. Since $R, N \perp P$, by part 6 of Theorem 2.3, $d, z \in P$ exist with $d := (R \cdot P)$ and $z := (P \cdot N)$. So, $R \cdot N = d \cdot z$. So, by part 3 of Theorem 2.3, $R \cdot N \cdot p = d \cdot z \cdot p \in P$ and $p \cdot R \cdot N = p \cdot d \cdot z \in P$. □

3 Reflection Spaces Related to Hyperbolic Planes

Let now $\mathcal{A} = (\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$ be a hyperbolic plane. Our main concern is to study the reflection spaces coming from the motion group $\mathfrak{M}$ of $\mathcal{A}$. We consider the involution set $\boldsymbol{J}$ and its specific subsets $\boldsymbol{E}$, representing point reflections, and $\mathfrak{G}$, representing line reflections. We study the *pairs* $(\boldsymbol{E}, < \boldsymbol{E} >), (\mathfrak{G}, < \mathfrak{G} >)$ and $(\boldsymbol{E} \cup \mathfrak{G}, < \boldsymbol{E} \cup \mathfrak{G} >) = (\boldsymbol{J}, \mathfrak{M})$.

3.1 Reflection Spaces Derived from Point Reflections

Firstly, we consider the pair $(\boldsymbol{E}, < \boldsymbol{E} >)$ consists of the set of point reflections $\boldsymbol{E}$.

Theorem 3.1 For any hyperbolic plane $\mathcal{A}$, $(\boldsymbol{E}, < \boldsymbol{E} >)$ is a polar-free reflection space and the corresponding incidence structure coincides with $(\boldsymbol{E}, \mathfrak{G})$.

Proof. (S1) is clear and **(S3)** follows from the fact that a hyperbolic plane is an ordinary absolute plane.

To verify **(S2),** we must show that for all $a, b \in \boldsymbol{E}$, $a \cdot b \cdot a \in \boldsymbol{E}$. Let $a, b \in \boldsymbol{E}$ with $a \neq b$, and let

$$L := \overline{a, b}, A := (a \perp L), B := (b \perp L).$$

So, by Theorem 2.3(6), $a = A \cdot L$ and $b = L \cdot B$. Then

$$a \cdot b \cdot a = (A \cdot L) \cdot (L \cdot B) \cdot (A \cdot L) = A \cdot B \cdot A \cdot L.$$

Since A and B are both perpendicular to L, we have $A, B \in L^{\perp}$. By Theorem 2.4(3) (Three Reflection Theorem for perpendicular lines), for any line $X \in \mathfrak{G}$: $A \cdot B \cdot X \in \boldsymbol{J} \Leftrightarrow X \perp L$. Since $A \perp L$, taking $X = A$ gives $A \cdot B \cdot A \in \boldsymbol{J}$. Since \$$A \cdot B \cdot A$ is a product of three line reflections, by the General Three Reflection Theorem (Theorem 2.4 part 1), $A \cdot B \cdot A = C$ for some line $C \in \mathfrak{G}$, and moreover $C \perp L$. Therefore: $a \cdot b \cdot a = C \cdot L$ with $C \perp L$. By part 6 of Theorem 2.3, the product of two mutually perpendicular lines is a point reflection, hence, $c := C \cdot L \in \boldsymbol{E}$. Thus, $a \cdot b \cdot a = c \in \boldsymbol{E}$.

We now show that $(\boldsymbol{E}, < \boldsymbol{E} >)$ is polar-free. Assume that there exist $a, x \in \boldsymbol{E}$ with $a \neq x$ such that $a \cdot x = x \cdot a$.

Let $L := \overline{a, x}, A := (a \perp L)$, and $X := (x \perp L)$. Since $a \neq x$, the corresponding perpendicular lines A and X are distinct. By part 6 of Theorem 2.3, we can write: $a = A \cdot L$ and $x = L \cdot X$. We have: $a \cdot x = A \cdot L \cdot L \cdot X = A \cdot X$ and $x \cdot a = X \cdot L \cdot L \cdot A = X \cdot A$. Hence, $a \cdot x = x \cdot a$ implies $A \cdot X = X \cdot A$. By Theorem 2.2(2) this implies $A \perp X$, hence by Theorem 2.3(6), $A \cap X \neq \emptyset$. However, since $A, X \perp L$, by part 7 of Theorem 2.3, $A \cap X = \emptyset$. This is a contradiction. Thus $a \cdot x \neq x \cdot a$ for all $a, x \in \boldsymbol{E}$, hence $(\boldsymbol{E}, < \boldsymbol{E} >)$ is polar-free. □

This result demonstrates that point reflections alone preserve the geometric structure. The polar-free property emerges naturally from the fact that distinct points never commute under multiplication, preventing the formation of non-trivial polars.

3.2 Reflection Spaces Derived from Line-Reflections

In this section we consider the pair $(\mathfrak{G}, \mathfrak{M})$. By part 1 of Theorem 2.3, we have $< \mathfrak{G} >= \mathfrak{M}$, hence **(S1)** holds. Let now $A, B \in \mathfrak{G}$, then we show the triple $A \cdot B \cdot A \in \mathfrak{G}$. We consider all possible geometric configurations between A and B:

1. A and B intersects i.e., $A \cap B \neq \emptyset$: Let $d \in A \cap B$, by Theorem 2.4(2), $A \cdot B \cdot X \in \boldsymbol{J}$ if and only if $d \in X$. Taking $X = A$, since $d \in A$, we have $A \cdot B \cdot A \in \boldsymbol{J}$. Moreover, since $A \cdot B \cdot A$ is a product of three line reflections, by the General Three Reflection Theorem (Theorem 2.4 part 1): $A \cdot B \cdot A = C \in \mathfrak{G}$. Hence, $A \cdot B \cdot A = C \in \mathfrak{G}$.
2. A and B have coperpendicular, i.e., $A^{\perp} \cap B^{\perp} \neq \emptyset$: Let $Y \in A^{\perp} \cap B^{\perp}$, i.e., $A, B \in Y^{\perp}$. By Theorem 2.4(3), $A \cdot B \cdot X \in \boldsymbol{J}$ if and only if $X \in Y^{\perp}$**.** Taking $X = A$, since $A \in Y^{\perp}$, we have $A \cdot B \cdot A \in \boldsymbol{J}$. Moreover, since $A \cdot B \cdot A$ is a product of three line reflections, by the General Three Reflection Theorem (Theorem 2.4 part 1): $A \cdot B \cdot A = C \in \mathfrak{G}$ with $C \in Y^{\perp}$. Hence, $A \cdot B \cdot A = C \in \mathfrak{G}$.
3. A and B are quasi parallel, i.e., $A \ddagger B$: In this case, by Theorem 2.4(4),), $A \cdot B \cdot X \in \boldsymbol{J}$ if and only if $X \in \widetilde{\widetilde{A, B}}$. Taking $X = A$, since $A \in \widetilde{\widetilde{A, B}}$, we have $A \cdot B \cdot A \in \boldsymbol{J}$. Moreover, since $A \cdot B \cdot A$ is a product of three line reflections, by the General Three Reflection Theorem (Theorem 2.4 part 1): $A \cdot B \cdot A = C \in \mathfrak{G}$ with $C \in \widetilde{\widetilde{A, B}}$. Hence, $A \cdot B \cdot A = C \in \mathfrak{G}$.

For any $A, B \in \mathfrak{G}$, $\widetilde{A, B} = \{X \in \mathfrak{G} | A \cdot B \cdot X \in \boldsymbol{J}\}$ is a line of $(\mathfrak{G}, \mathfrak{M})$. By Theorem 2.4, every such line is of one of the following three types:

1. **Point-pencils**: The set of lines passing through a given point $a \in \boldsymbol{E}$;

$$\mathfrak{G}(a) = \{X \in \mathfrak{G} | a \in X\}.$$

2. **Orthogonal pencil**: The set of lines perpendicular to a given line A;

$$A^{\perp} = \{X \in \mathfrak{G} | A \perp X\}.$$

3. **Quasi-ends**: The set of lines that are quasi-parallel A and B;

$$\widetilde{\widetilde{A, B}} = \{X \in \mathfrak{G} |\ A \ddagger X, B \ddagger X\}.$$

For a line $\widetilde{A, B}$ and $X, Y, Z \in \widetilde{A, B}$, by General Three Reflection Theorem (Theorem 2.4(1)), we have $X \cdot Y \cdot Z \in \widetilde{A, B}$**.** Therefore, **(S2)** is satisfied. Also, Axiom **(S3)** follows from Theorem 2.3(2).

Now, for any line $A \in \mathfrak{G}$ the polar is defined by $\Pi(A) = \{X \in \mathfrak{G} | A \cdot X = X \cdot A\}$. By Theorem 2.2, two line reflections commute if and only if the corresponding lines are perpendicular. Hence: $\Pi(A) = A^{\perp}$.

For any line $A \in \mathfrak{G}$, take any point $p \in A$. The unique line $(p \perp A)$ exists by the axioms of hyperbolic plane and belongs to $A^{\perp}$. Hence, $\Pi(A) \neq \emptyset$. Thus, $(\mathfrak{G}, \mathfrak{M})$ is polar.

We verify the polar axiom **(S4)** in its algebraic form: For all $A, X \in \mathfrak{G}$, $A \cdot X \cdot A = X \Longrightarrow X \in \Pi(A)$**.**

Assume that $A \cdot X \cdot A = X$. Then, $A \cdot X = X \cdot A$, so $X \in \Pi(A)$ by definition. By Theorem 2.2, this is equivalent to $A \perp X$, i.e., $X \in \Pi(A)$**.** Thus, the polar axiom **(S4)** holds.

Finally, by Theorem 2.3(5), any product of line reflections reduces to a product of two line reflections. Hence, $(\mathfrak{G}, \mathfrak{M})$ is reducible.

Thus, we obtain:

Theorem 3.2 *Let $\mathcal{A} = (\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$ be a hyperbolic plane. Then the pair $(\mathfrak{G}, \mathfrak{M})$ is a polar and reducible reflection space.*

Note that, the incidence structure of $(\mathfrak{G}, \mathfrak{M})$ is isomorphic to the dual of the incidence structure of the hyperbolic plane $(\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$ where the set $\mathfrak{G}$ of lines is structured by the pencils.

3.3 Reflection Spaces Derived from the Union of Point Reflections and Line Reflections

Finally, we study the pair $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ which satisfies (by part 1 of Theorem 2.3) **(S1)**. For $\mu \in \mathfrak{M} \setminus \{id\}$ we set:

$$\hat{\mu} := \{p \in \boldsymbol{E} \mid \mu \cdot p \in \boldsymbol{J}\}, \qquad \underline{\mu} := \{Y \in \mathfrak{G} \mid \mu \cdot Y \in \boldsymbol{J}\},$$
$$\hat{\hat{\mu}} := \widehat{\mu} \cup \underline{\mu}$$

Remark 3.1. Throughout this section, we identify each line $A \in \mathfrak{G}$ with its set of incident points in $\boldsymbol{E}$. In particular, expressions such as $\hat{\mu} = A$ are understood as equalities of subsets of $\boldsymbol{E}$.

In order to verify if $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ is a reflection space, we prove:

Theorem 3.3 Let $\mathcal{A} = (\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$ be a hyperbolic plane. Then:

1. For $K \in \mathfrak{G}$: $\widehat{K} = K = \{x \in \boldsymbol{E} | x \in K\}$, $\underline{K} = K^{\perp}$, *and* $\hat{\hat{K}} = K \cup K^{\perp}$.

2. For $m \in \boldsymbol{E}$: $\widehat{m} = \emptyset$ and $\underline{m} = \widehat{\widehat{m}} = \mathfrak{G}(m)$.
3. For distinct points $a, n \in \boldsymbol{E}$: $\overset{\frown}{a \cdot n} = \overline{a,n}$, $\underline{a \cdot n} = \overline{a,n}^{\perp}$, and $\widehat{\widehat{a.n}} = \overline{a,n} \cup \overline{a,n}^{\perp}$ in which $\overline{a,n}$ is the unique line through the points a and n.
4. For $a \in \boldsymbol{E}$, $B \in \mathfrak{G}$, and $C := (a \perp B)$: $\overset{\frown}{a \cdot B} = C$, $\underline{a.B} = C^{\perp}$, and $\widehat{\widehat{a.B}} = C \cup C^{\perp}$.
5. *If* $\{d\} = F \cap N$ *then* $\widehat{\widehat{F.N}} = \hat{\hat{d}} = \mathfrak{G}(d)$.
6. If $F \ddagger N$ *then* $\widehat{\widehat{F.N}} = \overbrace{\overbrace{F,N}}$ is a quasi-end containing F *and* N.

Proof.

1. $L \in \underline{K} \Leftrightarrow K \cdot L \in \boldsymbol{J} \Leftrightarrow$ (by part 2 of Theorem 2.2) $L \in K^{\perp}$, so $\underline{K} = K^{\perp}$. In the same way: $p \in \widehat{K} \Leftrightarrow K \cdot p \in \boldsymbol{J} \Leftrightarrow$ (by part 1 of Theorem 2.2) $p \in K$, and so $\widehat{K} = K$. Consequently, $\widehat{\widehat{K}} = K \cup K^{\perp}$.

2. By part 3 of Theorem 2.2, $\widehat{m} := \{p \in \boldsymbol{E} \mid m \cdot p \in \boldsymbol{J}\} = \emptyset$ and by part 1 of Theorem 2.2, $\underline{m} = \{X \in \mathfrak{G} | m \cdot X \in \boldsymbol{J}\} = \mathfrak{G}(m)$, hence $\widehat{\widehat{m}} = \mathfrak{G}(m)$.

3. By part 3 of Theorem 2.3, $\overset{\frown}{a \cdot n} = \overline{a,n}$.

Now, let $A := (a \perp \overline{a,n})$, $N := (n \perp \overline{a,n})$. So, by Theorem 2.3(6), $a = A \cdot \overline{a,n}$ and $n = \overline{a,n} \cdot N$. Therefore, $a \cdot n = (A \cdot \overline{a,n}) \cdot (\overline{a,n} \cdot N) = A \cdot N$. Let now $X \in \underline{a \cdot n}$. By definition, $a \cdot n \cdot X \in \boldsymbol{J}$. We have:

$$a \cdot n \cdot X = (A \cdot \overline{a,n}) \cdot (\overline{a,n} \cdot N) \cdot X = A \cdot N \cdot X \in \boldsymbol{J}.$$

Since $A, N \in \overline{a,n}^{\perp}$, by Theorem 2.4(3), $A \cdot N \cdot X \in \boldsymbol{J}$ iff $X \perp \overline{a,n}$, i.e., $X \in \overline{a,n}^{\perp}$. Hence, $\underline{a \cdot n} = \overline{a,n}^{\perp}$, and consequently, $\widehat{\widehat{a.n}} = \overline{a,n} \cup \overline{a,n}^{\perp}$.

4. By definition, $\overset{\frown}{a \cdot B} = \{x \in E | a \cdot B \cdot x \in \boldsymbol{J}\}$. Let $A' := (a \perp C)$. By Theorem 2.3(6), $a = A' \cdot C$. Now, for $x \in \overset{\frown}{a \cdot B}$ let $X := (x \perp C)$ and $X' := x \cdot X$. Hence, $x = X \cdot X'$ and by Theorem 2.3(6), $X' \perp X$. Therefore, $A', B, X \perp C$, hence by part 3 of Theorem 2.4 there is a line $Y \in \mathfrak{G}$ with $Y = A' \cdot B \cdot X$ and $Y \perp C$, i.e., $y := C \cdot Y$ is a point on C. Now, we can write:

$$a \cdot B \cdot x = (C \cdot A') \cdot B \cdot (X \cdot X') = C \cdot Y \cdot X' = y \cdot X' \in \boldsymbol{J} \Leftrightarrow \text{by Theorem 2.3(6), } y \in X'$$

Hence, $y \in X' \cap C$. Now, by uniqueness of perpendicular line from y to X we have $X' = C$. Thus, $x \in \overset{\frown}{a.B} \Leftrightarrow x \in C$.

Let now $X \in \underline{a.B}$. So, $a.B \cdot X \in \boldsymbol{J}$. Then by Theorem 2.5(1), $a.B \cdot X \in \boldsymbol{J}$ iff $X^{\perp} \cap B^{\perp} \neq \emptyset$, hence, $X \cap B = \emptyset$. Let $L := X^{\perp} \cap B^{\perp}$. By Theorem 2.3(6), for $x \in L \cap X$ and $b \in L \cap B$, $x = L \cdot X$ and $b = B \cdot L$, i.e., $L = \overline{x,b}$. We can write:

$$a \cdot B \cdot X = a \cdot B \cdot (L \cdot L) \cdot X = a \cdot (B \cdot L) \cdot (L \cdot X) = a \cdot b \cdot x \in \boldsymbol{J}.$$

Hence, by part 3 of Theorem 2.3, $a \in \overline{x,b} = L$. By uniqueness of perpendicular line form a point a to a line B, we conclude that $L = C$. Hence, $X \in C^{\perp}$. Thus, $\underline{a.B} = C^{\perp}$, and consequently, $\widehat{\widehat{a \cdot B}} = C \cup C^{\perp}$.

5. By part 1 of Theorem 2.5, $\{d\} = F \cap N$ implies $\overset{\frown}{F \cdot N} = \emptyset$. Let now $X \in \underline{F.N}$, hence, $F \cdot N \cdot X \in \boldsymbol{J}$. By Theorem 2.4(2), $F \cdot N \cdot X \in \boldsymbol{J}$ iff $d \in X$, i.e., $X \in \mathfrak{G}(d)$. Therefore: $\underline{F.N} = \mathfrak{G}(d)$. Since $\overset{\frown}{F \cdot N} = \emptyset$, it follows $\widehat{\widehat{F.N}} = \underline{F.N} = \hat{\hat{d}} = \mathfrak{G}(d)$.

6. Let $F \ddagger N$. By Theorem 2.5(1), $F^{\perp} \cap N^{\perp} = \emptyset$, hence, $\overset{\frown}{F \cdot N} = \emptyset$. Let now $X \in \underline{F.N}$, hence, $F \cdot N \cdot X \in \boldsymbol{J}$. By Theorem 2.4(4), $F \cdot N \cdot X \in \boldsymbol{J}$ iff $X \in \overbrace{\overbrace{F,N}}$. Thus, $\underline{F.N} = \overbrace{\overbrace{F,N}}$ is the quasi-end determined by F and N. Since $\overset{\frown}{F \cdot N} = \emptyset$, it follows $\underline{F.N} = \widehat{\widehat{F.N}} = \overbrace{\overbrace{F,N}}$. □

Remark 3.2 By part 4 of Theorem 2.3, $\boldsymbol{J} = \boldsymbol{E} \dot{\cup} \mathfrak{G}$. Let $\alpha, \beta \in \boldsymbol{E} \dot{\cup} \mathfrak{G}$, hence $\alpha^2 = \beta^2 = 1$, *i.e.*, α and β are involutions. Therefore

$$(\alpha \cdot \beta \cdot \alpha)^2 = \alpha \cdot \beta \cdot \alpha \cdot \alpha \cdot \beta \cdot \alpha = \alpha \cdot \beta \cdot \beta \cdot \alpha = \alpha \cdot \alpha = 1$$

Hence, any triple conjugate $\alpha \cdot \beta \cdot \alpha$ is again an involution, i.e., $\alpha \cdot \beta \cdot \alpha \in \boldsymbol{E} \cup \mathfrak{G}$.

For the pair $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ we can form the following types of lines:

$\boldsymbol{L_1} := \left\{\overbrace{k,v} \middle| k, v \in \boldsymbol{E}, k \neq v\right\}$, $\boldsymbol{L_2} := \{\overbrace{K,V} \mid K, V \in \mathfrak{G}, K \neq V\}$, and $\boldsymbol{L}_3 := \{\overbrace{k,V} \mid k \in \boldsymbol{E}, V \in \mathfrak{G}\}$. We have:

Theorem 3.4 Let $\overbrace{m,n} \in \boldsymbol{L}_1$, $\overbrace{M,N} \in \boldsymbol{L}_2$ and $\overbrace{m,N} \in \boldsymbol{L}_3$. Then:

1. Let $L := \overline{m,n}$ be the line determined by m and n, $G := (m \perp L)$ and $N := (n \perp L)$. Then, $\overbrace{m,n} = \overbrace{G,N} \in \boldsymbol{L}_2$ and $\overbrace{m,n} = \widehat{\widehat{m.n}} = \overline{m,n} \cup \overline{m,n}^{\perp}$. Moreover, if $\xi, \eta, \zeta \in \overbrace{m,n}$, then $\xi \cdot \eta \cdot \zeta \in \boldsymbol{E} \cup \mathfrak{G}$.
2. If M and N have no coperpendicular, then $\overbrace{m,n} = \widehat{\widehat{M.N}} = \mathfrak{G}(d)$ with $d := M \cap N$. Also, $\overbrace{m,n} = \overbrace{\overbrace{M,N}}$ is a quasi-end if M and N are quasi-parallel. Moreover, if $\xi, \eta, \zeta \in \overbrace{M,N}$ then $\xi \cdot \eta \cdot \zeta \in \mathfrak{G}$.
3. $\overbrace{m,N} = \widehat{\widehat{M.N}} = (m \perp N) \cup (m \perp N)^{\perp}$. *Moreover, if* $\xi, \eta, \zeta \in \overbrace{m,n}$ *then* $\xi \cdot \eta \cdot \zeta \in \mathfrak{G} \cup \boldsymbol{E}$.

Proof.

1. The first part follows from Theorem 3.3(3). Let now $\xi, \eta, \zeta \in \widetilde{m,n} = \overline{m,n} \cup \overline{m,n}^{\perp}$. If $\xi, \eta, \zeta \in \overline{m,n}$, then by Theorem 2.3(3), $\xi \cdot \eta \cdot \zeta \in \boldsymbol{E}$ and so $\xi \cdot \eta \cdot \zeta \in \overline{m,n}$. If $\xi, \eta, \zeta \in \overline{m,n}^{\perp}$, by Theorem 2.4(3), $\xi \cdot \eta \cdot \zeta \in \overline{m,n}^{\perp}$. Let $\xi, \eta \in L$ and $\zeta \in L^{\perp}$. Then by (2) of Theorem 2.5, $\xi \cdot \eta \cdot \zeta \in \mathfrak{G}$. If $\xi \in L$ and $\eta, \zeta \in L^{\perp}$ then by part 3 of Theorem 2.5, $\xi \cdot \eta \cdot \zeta \in L$.

2. The first part is a result of parts 5 and 6 of Theorem 3.3, and if $\xi, \eta, \zeta \in \widetilde{M,N}$ then ξ, η, ζ are lines and by the Three Reflection Theorem, $\xi \cdot \eta \cdot \zeta$ is a line contained in $\widetilde{M,N}$.

3. If $\xi, \eta, \zeta \in (m \perp N)$, i.e., ξ, η, ζ are collinear points on $C := (m \perp N)$, then by Theorem 2.3(3), $\xi \cdot \eta \cdot \zeta \in \boldsymbol{E}$. If $\xi, \eta, \zeta \in C^{\perp}$, then by Theorem 2.4(3), $\xi \cdot \eta \cdot \zeta \in C^{\perp} \subseteq \mathfrak{G}$. If $\xi, \eta \in C$ and $\zeta \in C^{\perp}$, then by Theorem 2.5(2), $\cdot \eta \cdot \zeta \in \mathfrak{G}$. If $\xi \in C$ and, $\eta, \zeta \in C^{\perp}$, then by Theorem 2.5(3), $\xi \cdot \eta \cdot \zeta \in C \subseteq \boldsymbol{E}$. In all subcases $\xi \cdot \eta \cdot \zeta \in \boldsymbol{J} = \mathfrak{G} \cup \boldsymbol{E}$. □

Theorem 3.4 states that for the lines of $\boldsymbol{L}$ the Three Reflection Theorem (**S2**) is valid.

Theorem 3.5 $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ is a reflection space.

Proof. By parts 1 and 4 of Theorem 2.3, **(S1)** is valid and also, in Theorem 3.4 we proved **(S2)**. There are always three non-collinear points $m, x,$ and g in a hyperbolic plane. So, by definition of ordinary absolute planes, $m \cdot x \cdot g \notin \boldsymbol{J}$. Or, by Theorem 2.3(2), there are $R, S, T \in \mathfrak{G}$ *with* $R \cdot S \cdot T \notin \boldsymbol{J}$, hence there are three non-collinear points in this case. Thus, $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ satisfies the axiom **(S3)**. □

By Theorem 3.4, there are the following types of lines in $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$:

1. For the line $M \in \mathfrak{G}$ there is a line $M_l \in \boldsymbol{L}$ of the reflection space $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ incident with the points of M and the lines of $\mathfrak{G}$ perpendicular to M. Let $\boldsymbol{L}_{\mathfrak{G}}$ denote the set of lines of this type.
2. If $m \in \boldsymbol{E}$ is a point of the plane, then there is a line $L_m \in \boldsymbol{L}$ of the reflection space $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ incident with the points which are lines of the hyperbolic plane passing through the point m. Let $\boldsymbol{L}_E$ denotes the set of lines of this type.
3. If $M, N \in \mathfrak{G}$ be two quasi-parallel lines of the hyperbolic plane, then there is a line $(M \ddagger N)_q \in \boldsymbol{L}$ of the reflection space $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ incident with the lines of the quasi-end determined by M and N.

We represent the set of lines of the reflection space $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ determined by quasi-ends of the hyperbolic plane by $\boldsymbol{L}_q$. Then, $\boldsymbol{L}_1 = \boldsymbol{L}_3 = \boldsymbol{L}_{\mathfrak{G}}$ and $\boldsymbol{L}_E \cup \boldsymbol{L}_q = \boldsymbol{L}_2 \backslash \boldsymbol{L}_1$ is a disjoint union.

Lemma 3.1 $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ is a polar reflection space.

Proof. We consider the polars for the pair $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$:
If $q \in \boldsymbol{E}$ then the polar $\Pi(q) = \mathfrak{G}(q)$ of the point $q \in \boldsymbol{E} \cup \mathfrak{G}$ is a line of $\boldsymbol{L}_{\boldsymbol{E}}$. If $M \in \mathfrak{G}$ then the polar $\Pi(M) = M \cup M^{\perp}$ of the point $M \in \mathfrak{G}$ is a line of $\boldsymbol{L}_{\mathfrak{G}}$.
These two statements provide that for $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ the polar axiom **(S4)** is satisfied and it is a polar reflection space. □

Lemma 3.2 (**E**∪$\mathfrak{G}$, $\mathfrak{M}$) is a reducible reflection space.

Proof. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \boldsymbol{E} \cup \mathfrak{G}$. By part 6 of Theorem 2.3, each point $a \in \boldsymbol{E}$ can be written as a product of two perpendicular lines: $a = X \cdot Y$, $X \perp Y$. Hence the product $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4$ can be expressed as a product of line reflections. Depending on the number of point reflections among the α_i, this yields a product of at most eight line reflections.

We now apply Theorem 2.3(5) (Reduction Theorem) repeatedly to shorten this product. In particular, any product of line reflections can be reduced to a product of at most three line reflections. Finally, by Lemma 2.1, any product of three line reflections is of the form $A \cdot B \cdot C = L \cdot p$, for some point $p \in \boldsymbol{E}$ and line $L \in \mathfrak{G}$. Therefore, the original product $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4$ reduces to either:

i. a product of two lines $L \cdot M$, or
ii. a product of a point and a line $L \cdot p$.

In both cases, the result is a product of two elements of $\boldsymbol{E} \cup \mathfrak{G}$. In detail:

1. $A \cdot B \cdot C \cdot D = U \cdot V$,
2. $A \cdot B \cdot C \cdot d = (A \cdot B \cdot C) \cdot (D \cdot E) = (A \cdot B \cdot C \cdot D) \cdot E = U \cdot V \cdot E = L \cdot p$,
3. $A \cdot B \cdot c \cdot D = A \cdot B \cdot (X \cdot Y) \cdot D = (A \cdot B \cdot X \cdot Y) \cdot D = U \cdot V \cdot D = M \cdot p$,
4. $A \cdot B \cdot e \cdot d = A \cdot B \cdot (X \cdot Y) \cdot (E \cdot F) = (A \cdot B \cdot X \cdot Y) \cdot (E \cdot F) = U \cdot V \cdot E \cdot F = M \cdot N$,
5. $A \cdot b \cdot C \cdot D = A \cdot (U \cdot V) \cdot C \cdot D = (A \cdot U \cdot V \cdot C) \cdot D = X \cdot Y \cdot D = L \cdot p$,
6. $A \cdot b \cdot C \cdot d = A \cdot (U \cdot V) \cdot C \cdot (X \cdot Y) = (A \cdot U \cdot V \cdot C) \cdot (X \cdot Y) = M \cdot N \cdot X \cdot Y = S \cdot T$,
7. $A \cdot b \cdot c \cdot D = A \cdot (U \cdot V) \cdot (X \cdot Y) \cdot D = (A \cdot U \cdot V \cdot X) \cdot Y \cdot D = M \cdot N \cdot Y \cdot D = S \cdot T$,
8. $A \cdot b \cdot c \cdot d = A \cdot (U \cdot V) \cdot (X \cdot Y) \cdot (S \cdot T) = (A \cdot U \cdot V \cdot X) \cdot Y \cdot (S \cdot T) = L \cdot M \cdot Y \cdot S \cdot T = W \cdot Z \cdot T = L \cdot p$,
9. $a \cdot B \cdot C \cdot D = (X \cdot Y) \cdot B \cdot C \cdot D = (X \cdot Y \cdot B \cdot C) \cdot D = U \cdot V \cdot D = L \cdot p$,
10. $a \cdot B \cdot C \cdot d = (X \cdot Y) \cdot B \cdot C \cdot (U \cdot V) = (X \cdot Y \cdot B \cdot C) \cdot (U \cdot V) = M \cdot N \cdot U \cdot V = S \cdot T$,
11. $a \cdot B \cdot c \cdot D = (X \cdot Y) \cdot B \cdot (U \cdot V) \cdot D = (X \cdot Y \cdot B \cdot U) \cdot V \cdot D = M \cdot N \cdot V \cdot D = S \cdot T$,
12. $a \cdot B \cdot c \cdot d = (X \cdot Y) \cdot B \cdot (U \cdot V) \cdot (M \cdot N) = (X \cdot Y \cdot B \cdot U) \cdot V \cdot (M \cdot N) = (S \cdot T) \cdot V \cdot M \cdot N = (S \cdot T \cdot V \cdot M) \cdot N = (F \cdot G) \cdot N = L \cdot p$,

13. $a \cdot b \cdot C \cdot D = (X \cdot Y) \cdot (U \cdot V) \cdot C \cdot D = (X \cdot Y \cdot U \cdot V) \cdot C \cdot D = M \cdot N \cdot C \cdot D = S \cdot T$,
14. $a \cdot b \cdot C \cdot d = (X \cdot Y) \cdot (U \cdot V) \cdot C \cdot (R \cdot S) = (X \cdot Y \cdot U \cdot V) \cdot C \cdot (R \cdot S) = (L \cdot M) \cdot C \cdot R \cdot S = (L \cdot M \cdot C \cdot R) \cdot S = (T \cdot N) \cdot S = L \cdot p$,
15. $a \cdot b \cdot c \cdot D = (X \cdot Y) \cdot (U \cdot V) \cdot (R \cdot S) \cdot D = (X \cdot Y \cdot U \cdot V) \cdot (R \cdot S) \cdot D = (L \cdot M) \cdot (R \cdot S) \cdot D = (L \cdot M \cdot R \cdot S) \cdot D = T \cdot M \cdot W = L \cdot p$,
16. $a \cdot b \cdot c \cdot d = (X \cdot Y) \cdot (U \cdot V) \cdot (R \cdot S) \cdot (F \cdot G) = (X \cdot Y \cdot U \cdot V) \cdot (R \cdot S \cdot F \cdot G) = (L \cdot M) \cdot (I \cdot J) = S \cdot T$.

Hence there exist $u, v \in \boldsymbol{E} \cup \mathfrak{G}$ such that $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 = u \cdot v$. Thus, ($\mathbf{E} \cup \mathfrak{G}$, $\mathfrak{M}$) is a reducible reflection space. □

As a direct consequence of Theorem 3.5, Lemmas 3.1, and Lemma 3.2 we obtain the following result regarding $(\boldsymbol{E} \cup \mathfrak{G}, \mathfrak{M})$:

Theorem 3.6 Let $\mathcal{A} = (\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$ be a hyperbolic plane. Then the pair $(\boldsymbol{E} \cup \mathfrak{G}, \mathfrak{M})$ is a reducible polar reflection space -called the maximal reflection space of $\mathcal{A}$- and so $(\boldsymbol{E} \cup \mathfrak{G}, \boldsymbol{L})$ is an incidence space.

Remark 3.3 Theorem 3.6 is fundamental for understanding hyperbolic planes through reflection theory. The reducibility property allows any product of four involutions to be expressed as a product of two, providing computational efficiency and enabling the study of the incidence structure in Section 4. The maximal reflection space unifies both point and line reflections, capturing the complete symmetry structure of the hyperbolic plane within a single algebraic framework.

4 Geometric Extensions and the Maximal Reflection Structure

In this section, we study the incidence structure $(\boldsymbol{E} \cup \mathfrak{G}, \boldsymbol{L})$ of the maximal reflection space $(\mathfrak{G} \cup \boldsymbol{E}, \mathfrak{M})$ related to a hyperbolic plane $\mathcal{A} = (\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$. In $\boldsymbol{L}$ there are three types of lines: $\boldsymbol{L_p} := \{\hat{\hat{x}} = \underline{x} = \mathfrak{G}(x) \mid x \in \boldsymbol{E}\}$ consists of point-pencils, $\boldsymbol{L_g} := \{\hat{\hat{X}} = X \cup X^{\perp} | X \in \mathfrak{G}\}$ arises from hyperbolic lines and their orthogonal complements, and $\boldsymbol{L_e} := \{\widehat{\widehat{X.Y}} = \underline{X \cdot Y} \mid X, Y \in \mathfrak{G}\ with\ X \ddagger Y\}$ consists precisely of those lines belonging to the quasi-end defined by X and Y. These lines encode the asymptotic directions of the hyperbolic plane.

Hence, each point $p \in \boldsymbol{E}$ of the hyperbolic plane determines a line of $\boldsymbol{L_p}$, the pencil $\mathfrak{G}(p)$. Also, each line $X \in \mathfrak{G}$ of the hyperbolic plane defines a corresponding line $\hat{\hat{X}}$ of $\boldsymbol{L_g}$ in the new structure, comprising all points and lines of the original plane that are perpendicular to it. Since there are quasi-parallel lines, there are also lines of type $\boldsymbol{L_e}$. This trichotomy reflects a fundamental principle: each geometric configuration in the hyperbolic plane (point, line, or quasi-end) generates a corresponding line in the reflection space. The interaction between these three types of lines determines the complete incidence structure.

Definition 4.1. A hyperbolic plane is called *complete* if any two distinct quasi-ends can be joined by a unique line. By Bachmann [5] (§16,3, p. 211), this is equivalent to the Axiom of Continuity (Dedekind's Axiom), under which the coordinate field is isomorphic to ℝ.

Theorem 4.1 (**Intersections of Lines**) For pairs of distinct lines $L_1 = \mathfrak{G}(r)$, $L_2 = \mathfrak{G}(n)$ in $\boldsymbol{L_p}$, $\boldsymbol{L_3} = T \cup T^{\perp}$, $L_4 = N \cup N^{\perp}$ in $\boldsymbol{L_g}$, and $\boldsymbol{L_5} = \underline{R \cdot S}$, $L_6 = \underline{X \cdot Y}$ in $\boldsymbol{L_e}$ (hence $R \ddagger S$ and $X \ddagger Y$) we have:

1. $L_1 \cap L_2 = \mathfrak{G}(r) \cap \mathfrak{G}(n) = \{\overline{r, n}\}$.
2. $L_3 \cap L_4 = (T \cup T^{\perp}) \cap (N \cup N^{\perp}) = \emptyset \Leftrightarrow T \ddagger N$.
3. $L_1 \cap L_3 = \mathfrak{G}(r) \cap (T \cup T^{\perp}) = \{(r \perp T)\}$.
4. $L_1 \cap L_5 = \mathfrak{G}(r) \cap \underline{R \cdot S}$ consists of exactly one line $W \in \mathfrak{G}$ (by 18.6 of [13]).
5. $L_3 \cap L_5 = (T \cup T^{\perp}) \cap \underline{R\ \cdot\ S} = T^{\perp} \cap \underline{R \cdot S}$.
6. $L_5 \cap L_6$ is a unique line if the hyperbolic plane is complete.

Proof. 1. Any line in $L_1 \cap L_2$ must pass through both r and n. By the incidence axioms of the hyperbolic plane, exactly one such line exists, namely $\overline{r, n}$.

2. Note that $(T \cup T^{\perp}) \cap (N \cup N^{\perp}) = (T \cap N) \cup (T^{\perp} \cap N^{\perp})$. Therefore, $L_3 \cap L_4 = \emptyset \Leftrightarrow T \ddagger N$.

3. $Z \in \mathfrak{G}(r) \cap (T \cup T^{\perp}) = \mathfrak{G}(r) \cap T^{\perp} \Leftrightarrow r \in Z$ and $Z \in T^{\perp}$, hence, $r \in Z$ and $T \perp Z$. Thus $Z = (r \perp T)$.

5. We can simplify $(T \cup T^{\perp}) \cap \underline{R\ \cdot\ S}$ as $\left(T \cap \underline{R\ \cdot S}\right) \cup \left(T^{\perp} \cap \underline{R\ \cdot\ S}\right)$ which is equivalent to $T^{\perp} \cap \underline{R \cdot S}$.

6. This result follows from Definition 4.1.
□

Proposition 4.2 $(\mathfrak{G} \cup \boldsymbol{E}, \boldsymbol{L})$ is a plane.

Proof. By statements 1, 3, and 4 of Theorem 4.1, each line of $\boldsymbol{L_p}$ intersects other lines of $\boldsymbol{L}$, hence elements of $\boldsymbol{L_p}$ are projective lines of the incidence space $(\mathfrak{G} \cup \boldsymbol{E}, \boldsymbol{L})$. So, for $L \in \boldsymbol{L_p}$ and $\alpha \in \mathfrak{G} \cup \boldsymbol{E}$ with $\alpha \notin L$, $\mathfrak{G} \cup \boldsymbol{E} = \bigcup\{\widetilde{\alpha, \xi} \mid \xi \in L\} = \overline{\overline{\alpha \cup L}}$. Therefore, $(\mathfrak{G} \cup \boldsymbol{E}, \boldsymbol{L})$ is a plane. □

Since there are quasi-parallel lines in the hyperbolic plane, by statement 2 of Theorem 4.1, the lines $M \cup M^{\perp}$ and $N \cup N^{\perp}$ of $\boldsymbol{L_g}$ have an empty intersection. Hence, $(\mathfrak{G} \cup \boldsymbol{E}, \boldsymbol{L})$ cannot be a projective plane. However, as $(\mathfrak{G} \cup \boldsymbol{E}, \boldsymbol{L})$ is already a plane by Proposition 4.2, it can be seen as a projective plane with the oval $\mathcal{E}$ removed. We now show that extending the incidence structure by adding $\mathcal{E}$ as new points yields a projective plane.

Let $\mathcal{E}$ represents the set of all quasi-ends. By Theorem 4.1 part 2, the only pairs of lines that fail to intersect are lines of $\boldsymbol{L_g}$ when $M \ddagger N$. Therefore, we only need to extend lines of this type.

We extend the incidence space $(\mathfrak{G} \cup \boldsymbol{E}, \boldsymbol{L})$ as follows:

Extended point set: $\Phi := \boldsymbol{E} \cup \mathfrak{G} \cup \mathcal{E}$.

Extended lines:

- $\boldsymbol{L_p} = \{\mathfrak{G}(x) \mid x \in \boldsymbol{E}\}$-unchanged.
- $\boldsymbol{L_e} = \mathcal{E} = \left\{\widetilde{\widetilde{A,B}} \,\middle|\, A, B \in \mathfrak{G} \; with \; A \ddagger B\right\}$-unchanged, each quasi-end ε is both a line and a new point of Φ.
- $\overline{\boldsymbol{L}}_{\boldsymbol{g}} = \{\bar{A} = A \cup A^{\perp} \cup \{\varepsilon_1, \varepsilon_2\} \mid A \in \mathfrak{G}\}$ -extended by the two quasi-ends $\varepsilon_1, \varepsilon_2$ containing A, which exist uniquely by Corollary 5.1 of [17].

The extended line set is $\overline{\boldsymbol{L}} := \overline{\boldsymbol{L}}_{\boldsymbol{g}} \cup \boldsymbol{L_p} \cup \boldsymbol{L_e}$. The incidence relation is defined as follows:

For $\alpha \in \boldsymbol{E} \cup \mathfrak{G}$ and a line $L \in \overline{\boldsymbol{L}}$ incidence is as in the original structure. A quasi-end $\varepsilon \in \mathcal{E}$ is incident with a line $A \in \mathfrak{G}$ if and only if $A \in \varepsilon$. Consequently, ε is incident with $\bar{A} \in \overline{\boldsymbol{L}}_{\boldsymbol{g}}$ if and only if $A \in \varepsilon$. Each quasi-end ε is incident with itself as a line in $\boldsymbol{L_e}$.

Theorem 4.3 Let $\mathcal{A} = (\boldsymbol{E}, \mathfrak{G}, \alpha, \equiv)$ be a complete hyperbolic plane and $\mathcal{E}$ be the set of all quasi-ends. Then:

1. The incidence space $(P, \boldsymbol{L})$ of the maximal reflection space $(\mathfrak{G} \cup \boldsymbol{E}, \boldsymbol{L})$ extends to a projective plane $(\Phi, \overline{\boldsymbol{L}})$.

2. The hyperbolic lines $\mathfrak{G}$ are trace structure of the projective plane on $\boldsymbol{E}$, i.e., $\mathfrak{G}=\{\boldsymbol{E} \cap \overline{\boldsymbol{L}} \mid \overline{L} \in \overline{\boldsymbol{L}},\; \boldsymbol{E} \cap \overline{L} \neq \emptyset\}$.

3. $\mathcal{E}$ is an oval in $(\Phi, \overline{\boldsymbol{L}})$.

Proof.

1. We show $(\Phi, \overline{\boldsymbol{L}})$ is a projective plane. We verify the three projective plane axioms:

Axiom P1: Any two distinct points determine a unique line.

Since $(\mathfrak{G} \cup \boldsymbol{E}, \boldsymbol{L})$ is already a plane by Proposition 4.2, any two distinct points of $\mathfrak{G} \cup \boldsymbol{E}$ already determine a unique line in $\boldsymbol{L} \subseteq \overline{\boldsymbol{L}}$. Therefore, we only need to verify the cases involving new points $\varepsilon \in \mathcal{E}$:

Case 1: $x \in \boldsymbol{E}$ and $\varepsilon \in \mathcal{E}$.

By 18.6 of [13], there exists a unique line $W \in \mathfrak{G}$ such that $x \in W$ and $W \in \varepsilon$. By Corollary 5.1 of [17], W belongs to exactly two quasi-ends ε_1 and ε_2 with $\varepsilon \in \{\varepsilon_1, \varepsilon_2\}$. The unique line through x and ε is $W \cup W^{\perp} \cup \{\varepsilon_1, \varepsilon_2\} \in \overline{\boldsymbol{L}}_g$.

Case 2: $X \in \mathfrak{G}$ and $\varepsilon \in \mathcal{E}$.

By part 5 of Theorem 4.1, $X^{\perp} \cap \varepsilon \neq \emptyset$. Hence, 18.6 of [13], there exists a unique line $W \in \mathfrak{G}$ such that $W \in \varepsilon$ and $W \perp X$, hence $X \in W^{\perp}$. By Corollary 5.1 of [17], W belongs to exactly two quasi-ends ε_1 and ε_2 with $\varepsilon \in \{\varepsilon_1, \varepsilon_2\}$. The unique line through X and ε is $W \cup W^{\perp} \cup \{\varepsilon_1, \varepsilon_2\} \in \overline{\boldsymbol{L}}_g$.

Case 3: $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$ with $\varepsilon_1 \neq \varepsilon_2$**.** Follows from Definition 4.1.

Axiom P2. Any two distinct lines meet.

By Theorem 4.1 parts 1–5, all pairs of lines in $(\mathfrak{G} \cup \boldsymbol{E}, \boldsymbol{L})$ already intersect except lines $T \cup T^{\perp}$ and $N \cup N^{\perp}$ when $T \ddagger N$. After extension, we only need to check this remaining case. Since $T \ddagger N$ by Theorem 2.4 part 4, T and N belong to a common quasi-end $\widetilde{\widetilde{T,N}}$. By Corollary 5.1 of [17], T belongs to exactly two quasi-ends $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$ and N belongs to exactly two quasi-ends $\varepsilon_3, \varepsilon_4 \in \mathcal{E}$, corresponding to their two asymptotic directions. Since $T \ddagger N$, they share exactly one common quasi-end:

$$\{\varepsilon_1, \varepsilon_2\} \cap \{\varepsilon_3, \varepsilon_4\} = \widetilde{\widetilde{T,N}}$$

where $\varepsilon_2 \neq \varepsilon_4$ correspond to the other asymptotic directions of T and N respectively. Therefore:

$$(T \cup T^{\perp} \cup \{\varepsilon_1, \varepsilon_2\}) \cap (N \cup N^{\perp} \cup \{\varepsilon_3, \varepsilon_4\}) \supseteq \widetilde{\widetilde{T,N}} \neq \emptyset.$$

Hence, any two distinct lines of $\overline{\boldsymbol{L}}$ meet.

Axiom P3. There exist at least four points of which no three are collinear.

Since $\mathcal{A}$ is a hyperbolic plane, there exist four points $p, q, r, s \in \boldsymbol{E}$ in general position, i.e., no three of which are collinear. Since the lines of $(\Phi, \overline{\boldsymbol{L}})$ restricted to $\boldsymbol{E}$ coincide with the lines of the hyperbolic plane, these four points remain non-collinear in $(\Phi, \overline{\boldsymbol{L}})$.

Therefore, $(\Phi, \overline{\boldsymbol{L}})$ is a projective plane.

2. Let $\overline{\boldsymbol{L}} \in \overline{\boldsymbol{L}}$**.** By the classification of projective lines, we have: $\overline{\boldsymbol{L}} = \overline{\boldsymbol{L}}_{\boldsymbol{g}} \cup \boldsymbol{L_p} \cup \boldsymbol{L_e}$. We consider each type:

- $\overline{\boldsymbol{L}} \in \boldsymbol{L_p}$: Then $\overline{L} = \mathfrak{G}(x)$ for some $x \in \boldsymbol{E}$. Its elements are line reflections, hence: $\overline{L} \cap \boldsymbol{E} = \emptyset$.
- $\overline{\boldsymbol{L}} \in \boldsymbol{L_e}$: Then $\overline{\boldsymbol{L}}$ is a quasi-end, i.e., an element of $\mathcal{E}$. So, $\overline{L} \cap \boldsymbol{E} = \emptyset$.
- $\overline{\boldsymbol{L}} \in \overline{\boldsymbol{L}}_{\boldsymbol{g}}$: Then $\overline{\boldsymbol{L}} = L \cup L^{\perp} \cup \{\varepsilon_1, \varepsilon_2\}$ with $L \in \mathfrak{G}$. By Remark 3.1, we identify each line $L \in \mathfrak{G}$ with its set of incident points in $\boldsymbol{E}$. Thus, L contributes exactly its incident points in $\boldsymbol{E}$, $L^{\perp} \subseteq \mathfrak{G}$ contains no points of $\boldsymbol{E}$, and $\{\varepsilon_1, \varepsilon_2\} \subseteq \mathcal{E}$ contains no points of $\boldsymbol{E}$. Hence, $\overline{L} \cap \boldsymbol{E} \neq \emptyset$.

Thus, we have shown: $\forall \bar{L} \in \bar{\boldsymbol{L}} : \bar{L} \cap \boldsymbol{E} \neq \emptyset \Leftrightarrow \bar{L} \in \bar{\boldsymbol{L}}_{\boldsymbol{g}}$. And, in this case $\bar{L} \cap \boldsymbol{E} = L \in \mathfrak{G}$.

Conversely, every $L \in \mathfrak{G}$ arises from some $\bar{L} \in \bar{\boldsymbol{L}}_{\boldsymbol{g}}$. Therefore, $\mathfrak{G}=\{\boldsymbol{E} \cap \bar{L} | \bar{L} \in \bar{\boldsymbol{L}},\ \boldsymbol{E} \cap \bar{L} \neq \emptyset\}$.

3. We verify the two conditions of the oval definition:

Condition 1: $\forall \bar{L} \in \bar{\boldsymbol{L}} : |\bar{L} \cap \mathcal{E}| \leq 2$. Remember $\bar{\boldsymbol{L}} = \bar{\boldsymbol{L}}_{\boldsymbol{g}} \cup \boldsymbol{L}_{\boldsymbol{p}} \cup \boldsymbol{L}_{\boldsymbol{e}}$. We verify the following three cases:

i. $\bar{L} \in \boldsymbol{L}_{\boldsymbol{p}}$: Then $\bar{L} = \mathfrak{G}(x)$ for some $x \in \boldsymbol{E}$. But elements of $\mathfrak{G}(x)$ are lines and elements of $\mathcal{E}$ are quasi-ends of mutually quasi-parallel lines, $\bar{L} \cap \mathcal{E} = \emptyset, i.e., |\bar{L} \cap \mathcal{E}| = 0$ in this case.

ii. $\bar{L} \in \boldsymbol{L}_{\boldsymbol{e}}$: Then $\bar{L}$ is a quasi-end, i.e., $\bar{L} = \varepsilon$ is exactly one element of $\mathcal{E}$. So, $\bar{L} \cap \mathcal{E} = \varepsilon, i.e., |\bar{L} \cap \mathcal{E}| = 1$.

iii. $\bar{\boldsymbol{L}} \in \bar{\boldsymbol{L}}_{\boldsymbol{g}}$: Then $\bar{\boldsymbol{L}} = L \cup L^{\perp} \cup \{\varepsilon_1, \varepsilon_2\}$ with $L \in \mathfrak{G}$. By Corollary 5.1 of [17], L belongs to exactly two quasi-ends ε_1 and ε_2. Therefore, $\bar{L} \cap \mathcal{E} = L \cup L^{\perp} \cup \{\varepsilon_1, \varepsilon_2\} = \{\varepsilon_1, \varepsilon_2\}, i.e., |\bar{L} \cap \mathcal{E}| = 2$.

Thus, $\forall \bar{L} \in \bar{\boldsymbol{L}} : |\bar{L} \cap \mathcal{E}| \leq 2$.

Condition 2: Through every $\varepsilon \in \mathcal{E}$ there passes exactly one tangent line.

A tangent to $\mathcal{E}$ at ε is a line $\bar{L} \in \bar{\boldsymbol{L}}$ with $\bar{L} \cap \mathcal{E} = \varepsilon$. But, by part ii of Condition 1, such a line exists: for any ε consider the line $\varepsilon \in \boldsymbol{L}_{\boldsymbol{e}}$, since $\varepsilon \cap \mathcal{E} = \varepsilon$. The uniqueness is a consequence of the categorization in Condition 1. Since each quasi-end ε corresponds to exactly one line in $\boldsymbol{L}_{\boldsymbol{e}}$, the tangent line is unique.

Thus, $\mathcal{E}$ is an oval in $(\Phi, \bar{\boldsymbol{L}})$. □

5 Discussion and Conclusion

The present work establishes the first systematic study of reflection spaces associated with hyperbolic planes, and our results reveal a structural picture that differs fundamentally from the elliptic case studied by Karzel and Taherian [14]. In [14], point reflections in an elliptic space generate a *polar* reflection space whose incidence structure forms a projective space — a setting in which distinct points can commute and no quasi-ends exist ($\omega_{\mathcal{A}} = 0$, cf. [17]). By contrast, we show that in the hyperbolic setting point reflections generate a *polar-free* reflection space (Theorem 3.1), since distinct hyperbolic points never commute, and the incidence structure recovers the hyperbolic plane itself rather than a projective space. The three-level hierarchy we establish, polar-free (point reflections), polar-reducible (line reflections), and maximal polar-reducible (union), has no complete analogue in the elliptic theory of [14] or in the general axiomatic framework of Karzel and Taherian [10, 11, 12].

Our results are consistent with, and complementary to, recent foundational developments in hyperbolic and absolute geometry. Augat and Hähl [19] recently presented a new axiom system for absolute plane geometry studying line reflections and motions, showing that the hyperbolic axiom recovers the classical algebraic models. While [19] proceeds from axioms toward the motion group, our work proceeds in the reverse direction, extracting reflection spaces from the involution structure of the motion group of a given hyperbolic plane. Bamberg and Penttila [20] characterized the hyperbolic plane as a pure incidence geometry, consistent with our Section 4 where the maximal reflection space embeds into a projective plane $(\Phi, \bar{\boldsymbol{L}})$ by adding quasi-ends as ideal points. Furthermore, Pambuccian, Struve, and Struve [21] demonstrated the power of reflection-geometric proofs, relying centrally on the Three-Reflection Theorem, in metric planes under minimal assumptions, supporting our choice to work with a general (non-continuous, non-Archimedean) hyperbolic plane throughout Sections 1–3.

The projective extension constructed in Theorem 4.3 also connects our work with the quasi-end classification of [17], where it is shown that for hyperbolic planes $\omega_{\mathcal{A}} = 2$, meaning every line belongs to exactly two quasi-ends. This invariant is geometrically realized in our result: the two quasi-ends per line correspond precisely to the two points in which each extended line meets the oval $\mathcal{E}$ in $(\Phi, \bar{\boldsymbol{L}})$.The criteria for quasi-parallelism established in [18] provide the technical foundation for the case analysis in Theorem 3.3, and the present work in turn shows that quasi-parallelism has a natural structural role in the reflection-space hierarchy of the hyperbolic plane.

In conclusion, we showed that point reflections generate a polar-free reflection space whose incidence structure coincides with the hyperbolic plane (Theorem 3.1); line reflections form a polar and reducible reflection space isomorphic to the dual plane (Theorem 3.2); and the union of point and line reflections provides a maximal polar-reducible reflection space (Theorem 3.6). Assuming the Axiom of Continuity, the incidence structure of the maximal reflection space extends to a projective plane $(\Phi, \bar{\boldsymbol{L}})$ in which the set $\mathcal{E}$ of all quasi-ends forms an oval (Theorem 4.3). Future research may include extending these results to higher-dimensional hyperbolic spaces, determining reflection spaces for other classes of absolute planes, studying the associated kinematic structures (K-loops and fibrations) in the sense of [11], and investigating whether $(\Phi, \bar{\boldsymbol{L}})$ is Pappian, a question requiring a coordinatization argument following [6, 7, 8, 9] and left as an open problem.

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